**Algebra Qualifying Examination**

August 2016

Do either one of $nA$ or $nB$ for $1 \leq n \leq 5$. Justify all your answers.

**1A.** Let $V$ be a finite dimensional complex vector space. Let $A$ and $B$ be two linear endomorphisms of $V$ satisfying $AB - BA = B$. Let $\lambda$ be an eigenvector of $B$ and $v \in V$ an eigenvector for $\lambda$.

a) Prove that the subspace $W$ spanned by $v, Av, A^2v, \cdots$ is $B$-invariant. (Hint: Show that $BA^kv = \lambda(A^kv + \sum_{i=0}^{k-1} a_iA^iv)$ holds for some $a_i \in \mathbb{C}$ for any $k \geq 0$.)

b) Prove that $W$ is a subspace of the null space of $B$ in $V$. (Hint: Let $n = \dim W$. Then $v, Av, \cdots, A^{n-1}v$ form a basis of $W$. Show that $\lambda = 0$.) Consequently, there is a common eigenvector in $W$ for $A$ and $B$.

**1B.** Let $A$ be a complex m by m matrix and let $B$ be a complex n by n matrix. Show that the determinant of the Kronecker product of $A$ and $B$ is $\det(A)^n \det(B)^m$.

**2A.** Prove that a group of order 150 is not simple. (Hint: use the set $\Sigma$ of all Sylow 5–subgroups in $G$ and consider the permutation representation of $G$ on $\Sigma$ which sends $P$ to $gPg^{-1}$.)

**2B.** Suppose $p$ is a prime and $G$ is a finite group. A subgroup $K$ of $G$ is called a *normal p–complement* if $K \triangleleft G$ and there is a Sylow $p$–subgroup $P$ such that $K \cap P = 1$ and $KP = G$. Show that if $G$ has a normal $p$–complement then it is unique. Show that if $G$ is a nilpotent group then $p$-complements exist.

**3A.** Check if the ring $\mathbb{Z}[\sqrt{-6}] = \{a + b\sqrt{-6} \mid a, b \in \mathbb{Z}\}$ is a UFD.

**3B.** Let $R$ be a PID and $I$ a nonzero ideal of $R$. Show that there are only finitely many ideals of $R$ containing $I$. Show by example that this may not hold if $R$ is a UFD but not a PID.

**4A.** Let $F$ be a field and $\bar{F}$ an algebraic closure of $F$. Let $f(x, y)$ and $g(x, y)$ be polynomials in $F[x, y]$ such that g.c.d$(f, g) = 1$ in $F[x, y]$. Show that there are only finitely many $(a, b) \in \bar{F}^2$ such that $f(a, b) = g(a, b) = 0$. (Hint: Use the Euclidean algorithm.)

**4B.** Let $\epsilon$ be a complex, primitive 20-th root of unity. Determine all subfields of $\mathbb{Q}(\epsilon)$ and for each subfield determine a primitive element.

**5A.** Let $D$ be a PID, and $D^n$ the free module of rank $n$ over $D$. Prove that any submodule of $D^n$ is a free module of rank $m \leq n$. (Hint: you may use that $D$ is Noetherian and any matrix $A = (a_{ij})$ with $a_{ij} \in D$ can be diagonalized in the sense of Smith Normal Form.)

**5B.** Let $G$ be the group with presentation

$$\langle x, y, z, t | (xz)^2(yt)^2, (xt)^4(zy)^3, (xy)^4(zt)^2 \rangle.$$

Write the commutator factor group of $G$ as a direct product of cyclic groups.