ALGEBRA QUALIFYING EXAMINATION

AUGUST 2018

Do either one of nA or nB for $1 \le n \le 5$. Justify all your answers.

1A. Find $X \in \mathrm{GL}_4(\mathbb{Q})$ such that if

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix},$$

then $X^{-1}AX$ is a direct sum of Jordan blocks.

1B. Suppose $A \in \mathrm{GL}_n(\mathbb{C})$ has finite order. Show that there is a matrix $T \in \mathrm{GL}_n(\mathbb{C})$ such that $T^{-1}AT$ is a diagonal matrix.

2A. Prove that if G is a non-abelian group of order 8, then G is isomorphic to the dihedral group D_4 or the quaternion group Q.

2B. Suppose S is a set on which the alternating group A_4 acts transitively. Determine all possibilities for $|S|$.

3A. Let A be a commutative noetherian ring with 1. Let M be a noetherian A-module. Prove that every A-submodule of M is finitely generated.

3B. Let A be a commutative ring with 1 and suppose that A has a unique maximal ideal M. Prove that every element of A is either invertible or an element of M.

4A. Let $f(x) = (x^2 - 3)(x^3 - 3)$. Calculate the splitting field K of f over $\mathbb{Q}$, the group $\mathrm{Gal}(K/\mathbb{Q})$, and list all fields L such that $\mathbb{Q} \subseteq L \subseteq K$ and $[L : \mathbb{Q}] = 4$.

4B. If L is a finite extension of the field K and the characteristic of K does not divide $[L\colon K]$, then L is a separable extension of K.

5A. Let A be a commutative ring with 1. Let P and M be A-modules. Consider the A-module homomorphism

$$\phi_{P,M} \colon \operatorname{Hom}_A(P, A) \otimes_A M \to \operatorname{Hom}_A(P, M),$$

which is induced by the A-bilinear homomorphism:

$$\operatorname{Hom}_A(P, A) \times M \to \operatorname{Hom}_A(P, M) \colon (f, m) \mapsto (p \mapsto f(p)m).$$

Prove that $\phi_{P,M}$ is an isomorphism whenever P is finitely generated and projective.

5B. Let A be an abelian group generated by elements $a, b, c \in A$ that satisfy $2a + 4b + 2c = 2a + 10b + 8c = 0$. Determine the order of the torsion subgroup T of A and write T as a direct sum of cyclic groups. Furthermore also determine the number of homomorphisms of T to a cyclic group of order 4.