Analysis Qualifying Exam - August 2016

PLEASE SHOW ALL YOUR WORK

Problem 1. Let $f_n(x) \geq 0$ be continuous functions on $[0,1]$. Suppose that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$ and that for all x, $\lim_{n\to\infty} f_n(x) = 0$. Prove or disprove that f_n must converge uniformly to 0 on $[0,1]$.

Problem 2. Let (X, μ) be a σ-finite measure space. Prove that $\mu(X) < \infty$ if and only if $L^2(X,\mu) \subset L^1(X,\mu)$.

Problem 3. Let m be Lebesgue measure on $[0,1]$. Given $f \in L^2([0,1], m)$ define

$$Kf(x) = \frac{1}{x^{4/3}} \int_0^x f(t)dt.$$

(a) Show that there is a constant C such that $\|Kf\|_1 \leq C\|f\|_2$ for all $f \in L^2([0,1], m)$, i.e., K is a bounded operator from $L^2([0,1], m)$ to $L^1([0,1], m)$.

(b) Find the operator norm of K.

Problem 4. Let (X, μ) be a finite measure space. Let $f \in L^1(X,\mu)$. For $t \in \mathbb{R}$ define

$$g(t) = \int_X \cos(tf(x))d\mu(x)$$

Prove that $g(t)$ is differentiable for all t and that the derivative is a continuous function on $\mathbb{R}$.

Problem 5. Let f_n be absolutely continuous functions on $[a,b]$, $f_n(a) = 0$ for all n. Suppose f_n' is a Cauchy sequence in $L^1([a,b], m)$ where m is the Lebesgue measure. Show that there exists an absolutely continuous function f on $[a,b]$ such that $f_n \to f$ uniformly on $[a,b]$.

Problem 6. Let $f,\ f_k : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable functions such that $f_k \to f$ a.e. and there exists a Lebesgue integrable function g ($g \in L^1(\mathbb{R})$) such that

$$|f_k(x)| \leq g(x) \text{ a.e. for all } k.$$

The goal in this problem is to prove that $f_k \to f$ almost uniformly, i.e, for any $\delta > 0$ there exists $E \subset \mathbb{R}$ such that $m(E) < \delta$ and $f_k \to f$ uniformly on E^c. The measure m is Lebesgue measure. Let $X_0 = \{x : g(x) = 0\}$ and $X_n = \{x : |g(x)| \geq 1/n\}$ for $n \in \mathbb{N}$ so that $\mathbb{R} = \bigcup_{n=0}^{\infty} X_n$.

(a) Show that X_n has finite measure for all $n \geq 1$.

(b) Show that for any $\delta > 0$, there is a set E such that $m(E) < \delta$ and for all $n \geq 1$ the sequence f_k converges to f uniformly on $E^c \cap X_n$.

(c) Show that f_k converges to f uniformly on E^c.