ANALYSIS QUALIFYING EXAM

JANUARY 2021

Please show all of your work. GOOD LUCK!

(1) Calculate the limit
$$\lim_{n\to\infty} \sqrt{n} \int_0^\infty \frac{1-\cos x}{1+nx^6} dx\,.$$
Justify all steps.

(2) Show that
$$\int_0^\infty \frac{\sin(x)}{x} e^{-x}\, dx = \frac{\pi}{4}\,.$$
Hint: You may want to consider integrals of the function $f(x, y) = e^{-xy}\sin(x)$.

(3) Let $f(x)$ be a twice continuously differentiable real-valued function on the interval $[0, 1]$. Suppose that $f''(x) + xf(x) = 0$, $f'(0) = 0$, and
$$\int_0^1 f(x)dx = 0\,.$$
Prove that
$$|f(1)| \le \frac{\sqrt{5}}{2} ||f||_{L^2([0,1])}\,.$$

(4) Define a measure μ on the Borel sigma-algebra on $[0, 1]$ by the formula
$$\mu(X) = m(\{y \in [0, \pi] :\ \sin y \in X\})$$
where m is the Lesbegue measure. Prove that μ is absolutely continuous with respect to the Lesbegue measure and find the Radon-Nikodym derivative $d\mu/dm$.

(5) For each $n \in \mathbb{N}$, let $f_n : [0, \infty) \to \mathbb{R}$ be a continuous function. Suppose f_n converges uniformly to f on $[0, \infty)$. Define functions $g_n : [0, \infty) \to \mathbb{R}$ and $g : [0, \infty) \to \mathbb{R}$ by setting
$$g_n(x) = \int_0^x f_n(t)\, dt \quad \text{and} \quad g(x) = \int_0^x f(t)\, dt$$
for all $x \in [0, \infty)$.

a) Prove or disprove (with a counter example) the following: for each $0 \le a < b < \infty$, g_n converges uniformly to g on $[a, b]$.

b) Prove or disprove (with a counter example) the following: g_n converges uniformly to g on $[0, \infty)$.

(6) Let H be a Hilbert space, and let $\{e_j\}_{j=1}^{\infty}$ be an ortho-normal basis in H. Define

$$x_k = \frac{e_1 + e_2 + \cdots e_k}{\sqrt{k}}.$$

a) Does the sequence $\{x_k\}$ converge weakly? If it does then what is its weak limit?
b) Does it converge in norm?
Give detailed proofs.