GEOMETRY-TOPOLOGY QUALIFYING EXAM, JANUARY, 2016

Problem 1

Let U denote an open subset of $\mathbb{C}$. A function $f : U \to \mathbb{C}$ is complex analytic on U if the limit

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists for all $z_0 \in U$.

(a) Directly from this definition, derive the Cauchy-Riemann equations.

(b) Suppose, in addition, that f is C^1 on U. Use Stokes' theorem to prove that

$$\int_{\partial X} f(z)dz = 0,$$

where $dz = dx + idy$, and $X \subset U$ is a compact 2-manifold with (smooth) boundary.

Problem 2

(a) Explain why the set

$$X_c := \{(x, y, z) \in \mathbb{R}^3 : xy + z^2 = c\}$$

is an embedded submanifold of $\mathbb{R}^3$, provided that $c \neq 0$.

(b) Find a basis for the tangent space to X_8 (viewed as a subspace of $\mathbb{R}^3$) at the point $(1, 4, 2)$.

(c) Let $i : X_8 \to \mathbb{R}^3$ denote the inclusion. Compute

$$i^*(dx \wedge dy + xdy \wedge dz)$$

evaluated at the point $(1, 4, 2)$, in terms of the basis dual to the basis you found in (b).

Problem 3

Consider the vector field v on S^2 which is given by

$$v|_{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \begin{pmatrix} -zy \\ zx \\ 0 \end{pmatrix}$$

where we have identified $TS^2|_q$ with $q^\perp$, the orthogonal complement. Find an explicit expression for the flow of this vector field, and graph the trajectories.

Problem 4

Let $\iota : S^3 \to \mathbb{R}^4$ be the inclusion map, and consider the 3-form on $\mathbb{R}^4$ given by

$$\alpha = i_E(dV) = x_1 dx_2 \wedge dx_3 \wedge dx_4 - x_2 dx_1 \wedge dx_3 \wedge dx_4 + x_3 dx_1 \wedge dx_2 \wedge dx_4 - x_4 dx_1 \wedge dx_2 \wedge dx_3$$

where E denotes the 'Euler vector field,' i.e. $E|_x = x$. Let $\beta = \iota^* \alpha$.

(a) Evaluate $\int_{S^3} \beta$.

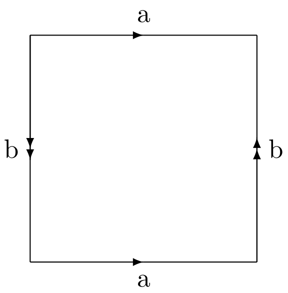
(b) Let γ be the following 3-form on $\mathbb{R}^4 \setminus \{0\}$:

$$\gamma = \frac{\alpha}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^k}$$

for $k \in \mathbb{R}$. Determine the values of k for which γ is closed and those for which it is exact.

Problem 5

The Klein bottle, B, can be obtained from a square by identifying its sides as indicated in the following figure:



In terms of this presentation, $\pi_1(B, o)$ is isomorphic to the group generated by a, b subject to the relation $aba^{-1}b = 1$, where o is the corner vertex (you do not need to prove this).

(a) Construct a model for the universal covering space of B.

(b) Fix a basepoint in your covering compatible with o. There is then an isomorphism of $\pi_1(B, o)$ with the group of automorphisms of the universal covering. Compute the actions of a and b in your model.

Problem 6

Calculate the integral homology of $\mathbb{RP}^3$.