Real Analysis Lectures – Integration workshop 2016

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Abstract

Lecture notes from the Integration Workshop at University of Arizona, August 2016. These notes borrow heavily from notes for previous workshops, written and revised by many people. There is more material than I can hope to cover in the lectures. I would urge you to go through all of it, even the parts that I don't get to in the lectures, because this background will be very useful in your core classes.

1 Lecture 1: Sequences and series

1.1 Definitions, convergence tests

Definition 1 *Let a_n be a sequence of real or complex numbers. We say that a_n* **converges** *to a if for every $\epsilon > 0$ there is an N such that $n \geq N$ implies $|a_n - a| < \epsilon$. We say that a_n is a Cauchy sequence if for every $\epsilon > 0$ there is an N such that $n, m \geq N$ implies $|a_n - a_m| < \epsilon$.*

Theorem 2 *The following statements are equivalent:*
(1) $\mathbb{R}$ is complete.
(2) Every bounded sequence in $\mathbb{R}$ has a least upper bound.

The second statement is taken as an axiom about $\mathbb{R}$. By the completeness axiom of the real numbers, a monotone sequence converges if and only if it is bounded. Given a sequence a_n, define $b_n = \sup\{a_k : k \geq n\}$. Then b_n is a nonincreasing sequence and so has a limit. Call this the limit superior or just lim sup a_n and write $\limsup_{n\to\infty} a_n$. Similarly, the limit inferior or lim inf is defined by

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k : k \geq n\}$$

The lim inf and lim sup always exist although the lim inf and the lim sup can be $\pm\infty$. We always have $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$. They are equal if and only if a_n converges. In this case they are equal to the limit of a_n and write

$$\lim_{n\to\infty} a_n.$$

Given a sequence a_n of real or complex numbers we can form the series $\sum_{n=1}^{\infty} a_n$. The partial sums are

$$s_n = \sum_{k=1}^{n} a_k$$

We can that the series converges to s if the partial sums s_n converges to s. We say it converges absolutely if $\sum_{n=1}^{\infty} |a_n| < \infty$.

Proposition 3 *(***Integral test***): Let f be a positive decreasing function defined on $[1, \infty)$ such that $lim_{x \to \infty} f(x) = 0$. For $n = 1, 2, \cdots$, define*

$$s_n = \sum_{k=1}^{n} f(k), \quad t_n = \int_1^n f(x)\,dx, \quad d_n = s_n - t_n$$

Then

- $0 < f(n+1) \leq d_{n+1} \leq f(1)$, *for* $n = 1, 2, \cdots$.
- $d = \lim_{n \to \infty} d_n$ *exists*
- s_n *converges if and only if* t_n *converges.*
- $0 \leq d_k - d \leq f(k)$ *for* $k = 1, 2, \cdots$.

Proposition 4 *(***Ratio and root tests***): Given a series $\sum_{n=1}^{\infty} a_n$ of nonzero complex terms, let*

$$r_- = \liminf_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|, \quad r_+ = \limsup_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|, \quad \rho = \limsup_{n \to \infty} |a_n|^{1/n}$$

- *The series converges absolutely if either $r_+ < 1$ or $\rho < 1$.*
- *The series diverges if either $r_- > 1$ or $\rho > 1$.*
- *In all other cases the tests are inconclusive.*

1.2 Infinite products

Let u_n be a sequence of complex numbers. The infinite product $\prod_{n=1}^{\infty} u_n$ is said to converge if there is an N such that $u_n \neq 0$ for $n \geq N$ and the sequence $p_k = \prod_{n=N+1}^{k}$ has a nonzero limit p as $k \to \infty$.

In the case of convergence, $\prod_{n=1}^{\infty} u_n$ is defined to be $pp_1 \cdots p_N$.

There is a connection betweeen convergence of sums and of products. For $a_n > 0$, the product $\prod_{n=1}^{\infty}(1 + a_n)$ converges if and only if the series $\sum_{n=1}^{\infty} a_n$ converges.

We say that the product $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely if $\prod_{n=1}^{\infty}(1+|a_n|)$ converges. Absolute convergence of the infinite product implies convergence of the product.

1.3 Sequences of functions

Definition 5 *A sequence of functions is said to* **converge pointwise** *to a limit function f on a set S provided that for every $x \in S$, and each $\epsilon > 0$, there exists N, depending possibly on both x and ϵ such that $n > N$ implies $|f_n(x) - f(x)| < \epsilon$. If the choice of N does not depend on x, the sequence of functions is said to* **converge uniformly.** *Let f_n be a sequence of functions defined on a set S. For each $x \in S$, set*

$$s_n(x) = \sum_{k=1}^{n} f_k(x)$$

If $s_n \to s$ uniformly on S, then we say that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on S.

Proposition 6 *(***Uniform convergence and continuity***) If $f_n \to f$ uniformly on S and each f_n is continouous at a point c, then f is continuous at c.*

Theorem 7 *(Weierstrass M-test) Let M_n be a sequence of nonnegative numbers such that $0 \leq |f_n(x)| \leq M_n$ for $n = 1, 2, \cdots$ and every $x \in S$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on S.*

The L^∞ norm

Consider the vector space $C(S)$, the real valued bounded continuous functions on a metric space S, and define the infinity norm (or sup norm) by

$$||f||_\infty = \sup_{x \in S} |f(s)|$$

Then $|| \quad ||_\infty$ is a norm This norm induces a metric $d(f, g) = ||f - g||_\infty$ in the usual way. The theorems above on uniform continuity show that $C(S)$ with this metric is a complete metric space.

Integration and differentiation

Many of the theorems on uniform convergence permit the reversal of the order of taking of limits.

Theorem 8 *(Integration) Let f_n be Riemann integrable functions on $[a, b]$ for $n = 1, 2, \cdots$. Define*

$$g_n(x) = \int_a^x f_n(t)dt, \quad x \in [a, b]$$

and

$$g(x) = \int_a^x f(t)dt, \quad x \in [a, b]$$

Assume there exists f so that $||f_n - f||_\infty \to 0$. Then

- *f is Riemann integrable and*
- $||g_n - g||_\infty \to 0$

Theorem 9 *(Differentiation) Assume that f_n is differentiable on (a, b), f_n' is Riemann integrable and that there exist a function g so that $d(f_n', g) \to 0$ and a point $c \in (a, b)$ so that $f_n(c)$ converges. Then*
(a) there exists f so that $d(f_n, f) \to 0$, and
(b) f is differentiable with derivative g.

2 Introduction to metric spaces

2.1 Topology of $\mathbb{R}^n$

For motivation, we recall what open and closed sets look like in $\mathbb{R}^n$. We use $|| \quad ||$ to denote the usual distance function in $\mathbb{R}^n$. A set $U \subset \mathbb{R}^n$ is open if for any $x \in U$ there is an $\epsilon > 0$ such that $||x - y|| < \epsilon$ implies $y \in U$. In other words, there is a ball centered at x that is entirely contained in U. A sequence x_n converges to x if $||x_n - x||$ goes to zero. A set F is closed if whenever a sequence x_n in F converges, the limit point must be in F.

If f is a function from an open subset U of $\mathbb{R}^n$ to $\mathbb{R}^k$, then f is continuous at x_0 if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $||x - x_0|| < \delta$ implies $||f(x) - f(x_0)|| < \epsilon$.

The only structure of $\mathbb{R}^n$ that we use in the above is the distance $||x - y||$ between two points. So we can abstract this by just starting with a set which has a distance function on it. Of course, the distance functions will need to have some properties.

2.2 Definition of a metric space and basic results

Definition 10 *A metric space (X, d) is a set X and a function (called the metric) $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$, the metric satisfies:*

1. *(positive definite) $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$*
2. *(symmetric) $d(x, y) = d(y, x)$*
3. *(triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$*

Definition 11 *In a metric space (X, d), a set U is said to be open if $\forall x \in U$, $\exists \epsilon > 0$ such that $d(y, x) < \epsilon$ implies $y \in U$.*

Proposition 12 *Let (X, d) be a metric space and let $\mathcal{T}$ be the collection of open sets in X. Then*

1. *$X \in \mathcal{T}$ and $\varnothing \in \mathcal{T}$,*
2. *Arbitrary unions of sets $U \in \mathcal{T}$ are in $\mathcal{T}$, i.e., for any indexing set I, if $U_i \in \mathcal{T}$ for all $i \in I$ then $\bigcup_{i \in I} U_i \in \mathcal{T}$,*

3. *If $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$.*

For any set X, a collection of subsets $\mathcal{T}$ of X is said to be a **topology** for X if it satisfies the three properties above. Note that property 3 immediately implies by induction that a finite intersection of open sets produces an open set.

Definition 13 *A sequence x_n in a metric space (X, d)* **converges** *to a point $x \in X$ if $\forall \epsilon > 0$, there exists an index $N < \infty$ such that $n > N \implies d(x_n, x) < \epsilon$.*

Definition 14 *A subset F of a metric space (X, d) is* **closed** *if for every sequence x_n in F which converges to some x in X we have $x \in F$.*

Proposition 15 *A set F is closed if and only if $F^C = X \setminus F$ is open.*

Corollary 16 *Arbitrary intersections and finite unions of closed sets are closed.*

Definition 17 *The interior of a set A, denoted $\overset{\circ}{A}$ or $int(A)$, is*

$$int(A) = \{x : \exists \epsilon > 0 \quad s.t. \quad d(x, y) < \epsilon \implies y \in A\}$$

The closure of a set A, denoted $\bar{A}$ or $cl(A)$, is

$$cl(A) = \{x : \forall \epsilon > 0 \quad \exists y \in A \quad s.t. \quad d(x, y) < \epsilon\}$$

Proposition 18 *The interior of a set A is the union of all open sets contained in A. The closure of a set A is the intersection of all closed sets containing A.*

Note that the proposition shows that the interior is open since it is a union of open sets, and the closure is closed since it is an intersection of closed sets.

2.3 Continuous maps

Definition 19 *Let (X, d) and (Y, d') be metric spaces and $f : X \to Y$ a function. For $x \in X$, we say f is continuous at x if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $d(x, y) < \delta$ implies $d'(f(x), f(y)) < \epsilon$. We say the map is globally continuous (or just continuous) if it is continuous at every point in X.*

Proposition 20 *Let (X, d) and (Y, d') be metric spaces and $f : X \to Y$ a function. Then f is continuous on X if and only if for every open set U in Y, $f^{-1}(U)$ is open.*

Show this proposition follows from the preceding definition! Note also that for a continuous function the inverse image of a closed set is closed.

Definition 21 *Let (X, d) and (Y, d') be metric spaces. A function $f : X \to Y$ is* **sequentially continuous** *if for every convergent sequence $x_n \to x$ in X, we have $f(x_n) \to f(x)$.*

Proposition 22 *In a metric space continuous = sequentially continuous.*

2.4 Additional material: Completeness and Uniform continuity

Definition 23 *A sequence x_n in a metric space is* **Cauchy** *if for any $\epsilon > 0$ there exist an integer N such that for $n, m \geq N$ we have $d(x_n, x_m) < \epsilon$. A metric space X is* **complete** *if every Cauchy sequence converges, i.e., for every Cauchy sequence x_n there is $x \in X$ such that x_n converges to x.*

Complete metric spaces are the natural setting for analysis, because, in these spaces Cauchy sequences (i.e sequences that "morally" should converge) are indeed convergent. They come up frequently in applications, because they allow one to assert the existence of a solution as the limit of an appropriately constructed sequence of approximate solutions. For this reason, it is useful to identify if a given metric space is complete, and if not (as for e.g $\mathbb{Q}$ is not complete), develop an approach to make a "natural" complete metric space (e.g. going from $\mathbb{Q}$ to $\mathbb{R}$.)

Completeness is **not** a topological property. Two metric spaces can be homeomorphic (topologically the same) while one if complete, and the other is not. The issue here is that continuous functions can map Cauchy sequences into sequences that are not Cauchy, and vice-versa.

A stronger notion than continuity is *uniform continuity.*

Definition 24 *A function $f : (M, d) \to (N, \rho)$ between metric spaces is* uniformly continuous *if for all $\epsilon > 0$, there is a $\delta > 0$ such that* for all $x \in M$, $d(y, x) < \delta \implies \rho(f(y), f(x)) < \epsilon$.

If f is uniformly continuous, then it will map Cauchy sequences into Cauchy sequence.

Proposition 25 *Every closed subset of a complete metric space is a complete metric space with the induced metric.*

This gives an approach to "completing" any metric space, that you will explore in the problem set.

3 Additional material: Compactness

3.1 Definitions of compactness

In a topological space there are two different notions of compact.

Definition 26 *A set F in a topological space $(X, \mathcal{T})$ is* **compact** *if for any collection of open sets whose union contains F (an open cover of F), there is a finite subcollection whose union still contains F (finite subcover).*

Definition 27 *A set F in a topological space $(X, \mathcal{T})$ is* **sequentially compact** *if every sequence contained in F has a limit point in F. That is, every sequence has a subsequence which converges to a point in F.*

Proposition 28 *In a metric space a set is compact if and only if it is sequentially compact.*

In a general topological space you can have sequentially compact sets which are not compact and compact sets which are not sequentially compact.

3.2 Properties of compact spaces

Proposition 29 *If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact.*

Proposition 30 *If $f : X \to \mathbb{R}$ is continuous and X is sequentially compact, then there exist x_{min} and x_{max} in X such that*

$$f(x_{min}) = \inf_{x \in X} f(x)$$
$$f(x_{max}) = \sup_{x \in X} f(x).$$

In words, f attains its minimum and maximum.

Proof. We prove that f attains its min. Let

$$m = \inf_{x \in X} f(x)$$

By the definition of inf, we can find $x_n \in X$ such that $f(x_n) < m + 1/n$. The sequence x_n has a subsequence x_{n_k} that converges to some $x_{min} \in X$. By continuity $f(x_{min}) = \lim_k f(x_{n_k})$. So we have

$$m \leq f(x_{min}) = \lim_k f(x_{n_k}) \leq \limsup_k [m + 1/n_k] = m$$

Hence $f(x_{min}) = m$. ■

3.3 Heine-Borel theorem

Notice that $(0,1]$, which is not closed, is not compact, because the cover $\bigcup_{k \in \mathbb{N}} \{(1/k, 1]\}$ has no finite subcover. And notice that $[0, \infty)$, which is not bounded, is not compact because the cover $[0, n]$, $n = 1, 2, \cdots$ has no finite subcover.

Definition 31 *A subset X of $\mathbb{R}^n$ is said to be* bounded *if there exists $r > 0$ such that $X \subset B(0, r) = \{x \in \mathbb{R}^n : |x| < r\}$.*

Theorem 32 *Subsets of $\mathbb{R}^n$ are compact if and only if they are closed and bounded.*

Since compactness and sequential compactness are equivalent in $\mathbb{R}^n$, we have

Theorem 33 *(Bolzano-Weierstrass) Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

3.4 Compactness in metric spaces

In $\mathbb{R}^n$, compactness is equivalent to being closed and bounded. This is not true in all metric spaces. There is a characterization of compactness in metric spaces. We have to replace boundedness by a stronger property and we also have to replace closed.

Definition 34 *In a metric space a set is totally bounded if $\forall \epsilon > 0$, the set can be convered by a finite number of balls of radius ϵ.*

Theorem 35 *A subset of a metric space is compact if and only if it is complete and totally bounded.*

4 Lecture 2: Calculus

4.1 Multivariate differential calculus

Definition 36 *Let $O \subset \mathbb{R}^n$ be open and $f : O \to \mathbb{R}^m$. Let $c \in O$. The function f is said to be* **differentiable** *at c if there is a linear function, called the* **total derivative**, *$T_c^f : \mathbb{R}^n \to \mathbb{R}^m$, such that*

$$\lim_{v \to 0} \frac{||f(c+v) - f(c) - T_c^f(v)||}{||v||} = 0 \tag{1}$$

We say f is differentiable on O if it is differentiable at every point in O.

If the total derivative exists, then for all directions the directional derivative exists

$$D_v f(c) = \lim_{\epsilon \to 0} \frac{f(c + \epsilon v) - f(c)}{\epsilon}$$

and equals $T_c^f(v)$.

Let $e_1, e_2, \cdots, e_n$ be the standard basis for $\mathbb{R}^n$. Then the partial derivatives are the directional derivative in the directions of the standard basis:

$$\frac{\partial f}{\partial x_k} = D_{e_k} f$$

Note that both sides of this equation are vectors in $\mathbb{R}^m$. The components are

$$\frac{\partial f_j}{\partial x_k} = D_{e_k} f_j$$

where $f = (f_1, f_2, \cdots, f_m)$. The matrix representation of T in this basis is called the **Jacobian matrix**

$$Df(c) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(c) & \frac{\partial f_1}{\partial x_2}(c) & \cdots & \frac{\partial f_1}{\partial x_n}(c) \\ \frac{\partial f_2}{\partial x_1}(c) & \frac{\partial f_2}{\partial x_2}(c) & \cdots & \frac{\partial f_2}{\partial x_n}(c) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{\partial f_m}{\partial x_1}(c) & \frac{\partial f_m}{\partial x_2}(c) & \cdots & \frac{\partial f_m}{\partial x_n}(c) \end{pmatrix}$$

Theorem 37 *(Chain rule) Suppose that* $f : O \subset \mathbb{R}^n \to \mathbb{R}^m$ *and* $g : U \subset \mathbb{R}^k \to O$ *are both differentiable. Then* $h = f \circ g$ *is differentable and*

$$T_p^h = T_{g(p)}^f \circ T_p^g$$

In matrix form

$$Dh(p) = Df(g(p))\, Dg(p)$$

Let $L(x_1, x_2) = \{\lambda x_1 + (1-\lambda)x_2 : 0 \leq \lambda \leq 1\}$ be the line segment connecting x_1 and x_2 in $\mathbb{R}^n$.

Theorem 38 *(Mean Value Theorem) Let* O *be an open subset of* $\mathbb{R}^n$ *and assume that* $f : O \to \mathbb{R}^m$ *is differentiable on* O*. Choose* x_1 *and* x_2 *so that* $L(x_1, x_2) \subset O$*. Then for every vector* $a \in \mathbb{R}^m$*, there is a point* $c \in L(x_1, x_2)$ *such that*

$$a \cdot (f(x_2) - f(x_1)) = a \cdot T_c^f(x_2 - x_1)$$

We now consider higher order derivatives.

Theorem 39 *Let* $f : \mathbb{R}^n \to \mathbb{R}^m$*. Then the following conditions are sufficient for the equality of the mixed partial derivatives*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(c) = \frac{\partial^2 f}{\partial x_j \partial x_i}(c)$$

1. Both $\partial f/\partial x_i$ *and* $\partial f/\partial x_j$ *exist in an* n*-ball* $B(c, \delta)$ *and are differentiable at* c*.*

2. Both $\partial f/\partial x_i$ *and* $\partial f/\partial x_j$ *exist in an* n*-ball* $B(c, \delta)$ *and* $\partial^2 f/\partial x_i \partial x_j$ *and* $\partial^2 f/\partial x_j \partial x_i$ *are both continuous at* c*.*

Call $\alpha = (\alpha_1, \cdots, \alpha_n)$ a multi-index if each of its entries are non-negative integers. Write $|\alpha| = \alpha_1 + \cdots + \alpha_n$. This allows for the notational abbreviations

$$x^\alpha = x^{\alpha_1} \cdots x^{\alpha_n}, \quad D^\alpha = \frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x^{\alpha_n}}$$

and provides for a compact notation for Taylors formula for functions f from $\mathbb{R}^n$ to $\mathbb{R}$. Write

$$f^{(k)}(x; t) = \sum_{\alpha : |\alpha| = k} D_\alpha f(x) t^\alpha$$

and assume that f and all of its partial derivatives of order up to $m-1$ are differentiable at each point of an open set $S \subset \mathbb{R}^n$. Choose x and a so that $L(a, x) \subset S$. Then for some $c \in L(a, x)$,

$$f(x) = f(a) + \sum_{k=1}^{m-1} \frac{1}{k!} f^{(k)}(x; t) + \frac{1}{m!} f^{(m)}(c; t)$$

4.2 Implicit functions

Let A be an $n \times n$ matrix. Then, for $y \in \mathbb{R}^n$, $Ax = y$ has a unique solution x whenever A has nonzero determinant. This suggests that in looking for a unique solution to $f(x) = y$, we consider the Jacobian determinant, the determinant of the Jacobian matrix,

$$J_f(x) = detDf(x) = \frac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)}$$

Theorem 40 *(Inverse function theorem) Let $f : S \to \mathbb{R}^n$ be continuously differentiable on an open set $S \subset \mathbb{R}^n$. If the Jacobian determinant $J_f(a) \neq 0$ for some point $a \in S$, then there exists two open sets $X \subset S$ and $Y \subset f(S)$ and a unique function g defined on Y such that*

1. *$a \in X$ and $f(a) \in Y$*
2. *$Y = f(X)$*
3. *f is one-to-one on X*
4. *$g(Y) = X$*
5. *$g(f(x)) = x$ for every $x \in X$*
6. *g is continuously differentiable on Y .*

Note that for $y = f(x)$, $Dg(y)\, Df(x)$ is the identity matrix.

Theorem 41 *(Implicit function theorem) Let $S \subset \mathbb{R}^n \times \mathbb{R}^k$ and suppose that $f : S \to \mathbb{R}^n$ is continuously differentiable. Assume that $f(x_0, y_0) = 0$ and that the $n \times n$ determinant $det[\partial f_j / \partial x_i(x_0, y_0)] \neq 0$. Then there exists a k-dimensional set Y_0 containing y_0 and a unique vector valued function $g : Y_0 \to \mathbb{R}^n$ such that*

1. *g is continuously differentiable*
2. *$g(y_0) = x_0$*
3. *$f(g(y), y) = 0$ for every $y \in Y_0$.*

4.3 Multivariable Riemann integrals

Let $A = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ and let $\mathcal{P}_k$ be a partition of $[a_k, b_k]$ into m_k intervals. Then

$$\mathcal{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$$

divides A into $m_1 \times \cdots \times m_n$ n dimensional intervals. A partition $\mathcal{Q}$ is called a refinement of $\mathcal{P}$ if $\mathcal{P} \subset \mathcal{Q}$.

For $I \in \mathcal{P}$ let $\mathrm{vol}(I)$ be the product of the lengths of the one dimensional intervals that determine I. Let f be a real valued function defined on A. For any choice of sample points $t_I \in I$, define the Riemann sum

$$S(f, \mathbf{t}, \mathcal{P}) = \sum_I f(t_I)\mathrm{vol}(I)$$

Theorem 42 *The function f is Riemann integrable on A if there exist a number, r, having the following property:*

For every $\epsilon > 0$, there exists a partition $\mathcal{P}_\epsilon$ such that for any refinement $\mathcal{Q}$ of $\mathcal{P}_\epsilon$ and any choice of sample points $t_I \in I$.

$$|S(f, \mathbf{t}, \mathcal{P}) - r| < \epsilon.$$

We typically write the integral

$$\int_A f(x_1, \ldots, x_n)\ d(x_1, \ldots, x_n) = \int_A f(x)\ dx.$$

4.4 Change of variable formulas for integrals

Let T be a one-to-one continuously differentiable mapping of an open set $V \subset \mathbb{R}^k$ into $\mathbb{R}^k$ such that the Jacobian determinant $J_T(x) \neq 0$ for all $x \in V$. Let f be a continuous function on $\mathbb{R}^k$ whose support is compact and lies in $T(V)$. Then

$$\int_{\mathbb{R}^k} f(y)\, dy = \int_{\mathbb{R}^k} f(T(x))\, |J_T(x)|\, dx$$

4.5 Differential forms and Stokes theorem

Let $K \subset \mathbb{R}^k$ be compact and let $V \subset \mathbb{R}^n$ be open. A k-surface is a continuously differentiable mapping $\Phi : K \to V$. For example, each component of a 1-surface is called a curve.

A *differential form of order* k, or briefly, a k-form, is a function ω, represented symbolically by

$$\omega = \sum a_{i_1 \cdots i_k}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

that assigns to each k-surface Ψ in V a number

$$\int_\Phi \omega = \int_K \sum a_{i_1 \cdots i_k}(\Phi(u))\, \frac{\partial(x_{i_1}, \cdots, x_{i_k})}{\partial(u_1, \cdots, u_k)} du$$

A 0-form is defined to be a continuous function of V . Integrals of 1-forms are called line integrals. Let $c \in \mathbb{R}$ and let $\omega, \omega_1, \omega_2$ be k-forms on V. Then

$$\int_\Phi c\omega = c \int_\Phi \omega$$

$$\int_\Phi (\omega_1 + \omega_2) = \int_\Phi \omega_1 + \int_\Phi \omega_2$$

For $\omega = a_{i_1 \cdots i_k}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ and for $\bar{\omega}$ obtained from ω by interchanging some pair of subscripts, $\bar{\omega} = -\omega$.

Write the basic k-form $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, for $1 \leq i_1 < \cdots < i_k$, giving the standard presentation

$$\omega = \sum_i a_I(x) dx_I$$

k	m	theorem
1	1	fundamental theorem
2	2	Green's theorem
3	3	divergence theorem
2	3	classical Stokes theorem

4.6 Differentiation of forms

The operator d is a mapping from k-forms to $(k+1)$-forms defined as follows:
1. For a class C^1 0-form f ,

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

2. For the class C^1 k-form ω above in the standard presentation,

$$d\omega = \sum_{I} (da_I) \wedge dx_I$$

For i = 1, 2, let ω_i be class C^1 k_i-forms. Then

$$d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge (d\omega_2)$$

If ω is of class C^2, $d(d\omega) = 0$.

Definition 43 *A k-form ω is called exact if $\omega = d\eta$ for some $(k-1)$-form η. A class C^1 k-form is called closed if $d\omega = 0$.*

Every exact class C^1 form is closed. If the domain is a convex set, then the Poincare lemma states that the converse is true.

Theorem 44 (General Stokes' theorem) *If Ψ is a k-chain of class C^2 in an open set $V \subset \mathbb{R}^m$ and if ω is a $(k-1)$-form of class C^1 in V , then*

$$\int_{\Psi} d\omega = \int_{\partial\Psi} \omega$$

Various theorems from calculus and vector calculus are special cases of this general theorem as indicated in the following table.

Theorem 45 (Green's Theorem) *Let C be a simple closed curve in the xy-plane Let $M(x, y)$ and $N(x, y)$ be continuously differentiable on an open set containing C and R, the region it encloses. Then*

$$\int_C (Mdx + Ndy) = \int_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}) dxdy$$

Theorem 46 (Divergence Theorem) *Let F be a continuously differentiable vector field on an open set $V \subset \mathbb{R}^3$, and let $C \subset V$ be closed with positively oriented boundary ∂C. Then*

$$\int_C (\nabla \cdot F) dV = \int_{\partial C} (F \cdot \mathbf{n})\, dA$$

where $\mathbf{n}$ *is a unit normal vector, pointing outwards.*

Theorem 47 (classical Stokes' Theorem) *Let F be a continuously differentiable vector field on an open set $V \subset \mathbb{R}^3$, and let $S \subset V$ be a 2-surface of class C^2. Then*

$$\int_S (\nabla \times F) \cdot \mathbf{n}\, dV = \int_{\partial S} (F \cdot \mathbf{t})\, ds$$

where $\mathbf{t}$ *is a oriented unit tangent vector.*

5 Lecture 3: Complex Analysis

5.1 Analytic functions

Let O be an open subset of $\mathbb{C}$. Let $f : O \to \mathbb{C}$. We say f is analytic at z_0 if the following complex limit exists:

$$f'(w) = \lim_{z \to w} \frac{f(z) - f(w)}{z - w}$$

for all w in a neighborhood of z_0. One can think of a function from $\mathbb{C}$ to $\mathbb{C}$ as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. We write $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$. Then the function is $F(x, y) = (u(x, y), v(x, y))$. It is important to understand that analyticity is a much stronger property than requiring that F have a total derivative. The above limit involves complex numbers and so it includes as special cases z approaching z_0 along any direction. These "directional limits" must all give the same complex number as the limit. In particular, by considering taking the limit in the coordinate directions, one obtains the Cauchy Riemann equations.

Theorem 48 *f is analytic at $z_0 = (x_0, y_0)$ if and only if for all (x, y) in a neighborhood of (x_0, y_0) the total derivative of F exists and*

$$\frac{\partial u}{\partial x}(x, y) = \frac{\partial v}{\partial y}(x, y), \quad \frac{\partial u}{\partial y}(x, y) = -\frac{\partial v}{\partial x}(x, y).$$

5.2 Power series

An infinite series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is called a power series centered at z_0. Define r by $1/r = \limsup_{r\to\infty} |a_n|^{1/n}$ (We make the conventions that $1/0 = \infty$ and $1/\infty = 0$.) Then by the root test, the series converges absolutely if $|z - z_0| < r$ and diverges if $|z - z_0| > r$. Furthermore:

1. The series converges uniformly on every compact subset of $B(z_0, r)$.
2. The function f can be differentiated term by term for any $z \in B(z_0, r)$,

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$$

3. The power series for f' has radius of convergence r.
4. Repeated differentiation and evaluation of this yields $a_k = f^{(k)}(z_0)/k!$.

Theorem 49 *Suppose that the power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

has a nonzero radius r of convergence. Then f is analytic on $B(z_0, r)$. Conversely, if f is analytic at z_0, then there is a power series with a nonzero radius of convergence that converges to f in a neighborhood of z_0.

5.3 Integration

Definition 50 *A domain D is simply connected if the region bounded by every simple closed curve in D is contained in D, i.e., every simple closed curve in D may be continuously contracted to a point without leaving D.*

Theorem 51 (one of many "Cauchy's theorems") *If D is a simply connected open set and f is analytic on D and γ is a differentiable closed curve in D, then*

$$\oint_{\gamma} f(z) dz = 0 \tag{2}$$

5.4 Zeroes, poles and residues

We say f has a zero at z_0 if $f(z_0) = 0$. In this case it is possible to write it in the form $f(z) = (z - z_0)^n g(z)$ in a neighborhood of z_0 where g does not vanish on this neighborhood. The integer n is unique and called the *order* of the zero.

A neigborhood of a point z_0 means an open set containing z_0. By a deleted neighborhood of z_0 we will mean a neighborhood of z_0 with z_0 removed. A function f has an isolated singularity at z_0 if it is analytic on a deleted neighborhood of z_0.

If f has an isolated singularity at z_0, and we can redefine it at z_0 so that the function is analytic at z_0, then we say f has a removable singularity at z_0. Otherwise we consider $1/f$ where $1/f$ is defined to be 0 at z_0. If this is anayltic at z_0 we say f has a pole at z_0. The order of the pole is defined as the order of the zero of $1/f$ at z_0. If it does not have a pole we say it has an essential singularity.

Theorem 52 *If f has a pole of order n at z_0 then*

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0} + g(z)$$

where g is analytic at z_0.

The *principal part* of $f(z)$ (at z_0) is

$$\frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0}$$

The *residue* of f at z_0 is a_{-1}