Analysis problem set – Integration workshop 2016

August 2, 2016

1 Sequences and series

1.1 Let x_n be a bounded sequence of real numbers. Let $x = \limsup_{n\to\infty} x_n$.

(a) Prove that there is a subsequence x_{n_k} such that $\lim_{k\to\infty} x_{n_k} = x$.

(b) Prove that for every subsequence x_{m_k} of x_n which converges, $\lim_{k\to\infty} x_{m_k} \leq x$

1.2 Assume that $a_n > 0$ and $b_n > 0$ for n = 1, 2, . . ., and suppose that $\lim_{n\to\infty} a_n/b_n = 1$. Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

1.3 Show that the product $\prod_n (1 + a_n)$ converges (resp. converges absolutely) if and only if the series $\sum a_n$ converges (resp. converges absolutely).

1.4 A non-negative sequence a_n is *subadditive* if $a_{n+m} \leq a_n + a_m$ for all $m, n \in \mathbb{Z}^+$. The goal of this problem is to show that, for any subadditive sequence, the limit $\lim_{n\to\infty} \frac{a_n}{n}$ exists.

(a) Show that, for any $n \in \mathbb{Z}^+$, there is a constant C_n such that, for all $k \geq n+1$, we have

$$\frac{a_k}{k} \leq \frac{a_n}{n} + \frac{C}{k}.$$

(b) Show that $\limsup_{k\to\infty} \frac{a_k}{k} \leq \liminf_{n\to\infty} \frac{a_n}{n}$. Why is this enough to prove the claim?

1.5 A_n is a sequence of subsets of $\mathbb{R}$.

(a) Explain why the following definitions make sense:

$$\limsup_n A_n = \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} A_n = \{x | x \in A_n \textit{ infinitely often}\}$$

and

$$\liminf_n A_n = \bigcup_{j=1}^{\infty} \bigcap_{n=j}^{\infty} A_n = \{x | x \in A_n \textit{ eventually}\}$$

(b) What can you conclude about a sequence of subsets for which $\limsup_n A_n = \liminf_n A_n$. Explain why this is a very restrictive definition of a limit for sets. In your classes you will explore alternative (and less restrictive) notions for limits of sets.

1.6 (a) Let f be a continuous function on $[0,1]$. Show that the following limits exist and evaluate them:

$$\lim_{n\to\infty} \int_0^1 x^n\, f(x)\, dx$$

$$\lim_{n\to\infty} n \int_0^1 x^n\, f(x)\, dx$$

(b) Let g be a differentiable function on $[0,1]$ such that $g(1) = 0$. Show that the following limit exists and evaluate:

$$\lim_{n\to\infty} n^2 \int_0^1 x^n\, g(x)\, dx$$

1.7 Consider the alternating series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

(a) Show that, as presented the series (i.e the sequence of partial sums taken with the given ordering) converges to $\ln(2)$.
(b) Show that, given any pair of real numbers $a < b$, the terms in the series can be rearranged so that the sequence x_n of partial sums (of the rearranged series) satisfies

$$\limsup_n x_n \geq b, \qquad \liminf_n x_n \leq a.$$

1.8 Let

$$f(x) = \begin{cases} a & x = 0 \\ x^{-x} & x > 0 \end{cases}$$

(a) Show that there is an appropriate choice for a so that f is continuous on $[0,1]$.

(b) For this choice of a, show, with justification, that

$$\int_0^1 f(x)dx = \sum_{n=1}^{\infty} n^{-n}.$$

Possibly useful integral: If $\beta > 0$ and $n \in \mathbb{N}$,

$$\int_0^\infty x^n e^{-\beta x} dx = \frac{n!}{\beta^{(n+1)}}$$

1.9 Give a counter example to the following statement: A sequence of differentiable functions f_n converges uniformly to a differentiable function f. This implies that the sequence of derivatives f_n' converges pointwise to f'.

1.10 f_n is a sequence of uniformly bounded non-negative Riemann integrable functions. For each $x \in [0,1]$, the real sequence $f_k(x)$ is monotone non-decreasing, i.e $n \geq m$ and $x \in [0,1]$ implies that $f_n(x) \geq f_m(x)$, and $0 \leq f_n(x) \leq K < \infty$ for all n, x.

(a) Show that the sequence $f_n(x)$ converges pointwise to a bounded function $f(x)$.

(b) Show that the sequence of real numbers $s_k = \int_0^1 f_k(x)\,dx$ also converges to some number s.

(c) Is it true that f is Riemann integrable, and $\int_0^1 f(x)\,dx = s$? Prove or give a counter example.

2 Metric spaces and continuous functions

2.1 For $x, y \in \mathbb{R}$ let

$$d(x,y) = \frac{|x-y|}{1+|x-y|}$$

(a) Show this is a metric.

(b) Does this metric give $\mathbb{R}$ a different topology from the one that comes from the usual metric on $\mathbb{R}$? You should prove your answer.

2.2 The structure of open sets in $\mathbb{R}$.

In what follows, we are considering the standard (metric) topology on $\mathbb{R}$.

(a) Let S be a nonempty open subset of $\mathbb{R}$. For each $x \in S$, let $A_x = \{a \in \mathbb{R} : (a, x] \subseteq S\}$ and $B_x = \{b \in \mathbb{R} : [x, b) \subseteq S\}$. Show that, A_x and B_x are both non-empty.

(b) Where $x \in S$ as above, if A_x is bounded below, let $a_x = \inf(A_x)$. Otherwise, let $a_x = -\infty$, and define b_x is a corresponding manner. Show that $x \in I_x = (a_x, b_x) \subseteq S$.

(c) Show that $S = \cup_x I_x$.

(d) Show that the intervals I_x give a partition of S, *i.e.*, for $x, y \in S$, either $I_x = I_y$ or $I_x \cap I_y = \emptyset$.

(e) Show that the set of distinct intervals $\{I_x : x \in S\}$ is countable.

(f) Prove that every open set in $\mathbb{R}$ is a countable disjoint union of open intervals.

2.3 If $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ and $g : (Y, \mathcal{S}) \to (Z, \mathcal{V})$ are continuous, show that the composition $g \circ f : (X, \mathcal{T}) \to (Z, \mathcal{V})$ is also continuous.

2.4 A subset A of X is called dense if the closure of A is X. A topological space X is called *separable* if there exists a countable dense subset.

(a) Prove that $\mathbb{R}^n$ with the usual topology is separable.

(b) Since the rationals are a dense set in $\mathbb{R}$ does it follow that every open subset of $\mathbb{R}$ is determined by the rational elements of the set?

(c) Show that the collection of all intervals of the form (r_1, r_2) where both r_1 and r_2 are rational is countable, and further, every open subset of $\mathbb{R}$ is uniquely determined by the intervals with rational endpoints that are contained in it.

(d) Is the collection of all open subsets of $\mathbb{R}$ countable?

(e) If a metric space is separable, show that there is a countable collection of open balls (i.e sets of the form $B_\epsilon(x) = \{y : d(x, y) < \epsilon\}$) such that every open set can be written as a union of balls from this collection.

2.5 Accumulation points $A \subseteq \mathbb{R}$, and A' denotes the set of all the accumulation points of A.

(a) If $y \in A'$ and $U \subseteq \mathbb{R}$ is an open set containing y, show that there are infinitely many distinct points in $A \cap U$.

(b) Show that

$$A' = \bigcap_{x \in A} \mathrm{cl}(A \backslash \{x\}).$$

(c) Using this, or otherwise, show that A' is a closed set.

(d) Show that $cl(A) = A \cup A'$.

2.6 (a) Show that $\mathbb{R}$ and $(0, 1)$ are homeomorphic, i.e., there is continuous bijection between them whose inverse is also continuous.

(b) Let $f : (0, 1) \to \mathbb{R}$ be your homeomorphism. Show there is a Cauchy sequence x_n in $(0, 1)$ such that $f(x_n)$ is not Cauchy in $\mathbb{R}$.

2.7 In the lectures we stated a proposition that said that if (X, d) and (Y, d') are metric spaces and $f : X \to Y$, then the $\epsilon - \delta$ definition of continuity of f and the open set defintion are equivalent. Prove this proposition.

2.8 Let $C((0, 1))$ be the set of bounded continuous functions on $(0, 1)$ with the usual sup norm. So

$$d(f, g) = \sup_{0<x<1} |f(x) - g(x)|$$

Let $U = \{f : f(x) > 0 \quad \forall x \in (0, 1)\}$, and $V = \{f : f(x) \geq 0 \quad \forall x \in (0, 1)\}$. For each of U and V determine if the set is open or closed or neither. You should prove your answer.

2.9 (a) Define a function f on $[0, 1]$ by

$$f(x) = \begin{cases} 0 & x \text{ is irrational} \\ 1 & x \text{ is rational} \end{cases}$$

Identify the points where f is continuous and the points where f is discontinuous.

(b) Define a function g on $[0, 1]$ by

$$g(x) = \begin{cases} 0 & x \text{ is irrational} \\ \frac{1}{q} & x = \frac{p}{q} \text{ expressed in its lowest terms} \end{cases}$$

Identify the points where f is continuous and the points where f is discontinuous.

(c) (Optional.. hard] Is there an example of a function $f : [0; 1] \to \mathbb{R}$ that is discontinuous at all the irrationals and continuous at all the rationals?

2.10 A function $f : \mathbb{R} \to \mathbb{R}$ is *uniformly continuous* if for all $\epsilon > 0$, there is a $\delta > 0$ such that *for all* $x \in \mathbb{R}$, $|y - x| < \delta \implies |f(y) - f(x)| < \epsilon$. This means we can pick a δ "that works" for all points in $\mathbb{R}$. Clearly, this definition also extends to general metric spaces.

(a) Let f be a continuous real valued function on $[0, \infty)$ such that $\lim_{x\to\infty} f(x)$ exists (and is finite). Prove that f is uniformly continuous on $[0, \infty)$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous. If x_n is a Cauchy sequence, show that $f(x_n)$ is also a Cauchy sequence. Is the converse true?

2.11 Let (M, d) be a metric space. Define a set $\tilde{M} \subset M^{\mathbb{N}}$ as the collection of all the Cauchy sequences in (M, d).

(a) Show that $\rho(\{x_n\}, \{y_n\}) = \lim\limits_{n\to\infty} d(x_n, y_n)$ is a well defined function on $\tilde{M} \times \tilde{M}$, i.e the limit always exists.

(b) Show that ρ satisfies the triangle inequality.

(c) Let $a_k = \{x_n\}_k$ be a sequence in $\tilde{M}$ (i.e a sequence of Cauchy sequences in (M, d)). Further, assume that a_k is a Cauchy sequence with respect to ρ, i.e. for all $\epsilon > 0$, there is an index K such that for all $j, k > K$, we have

$$\rho(a_j, a_k) = \lim_{n\to\infty} d((x_n)_j, (x_n)_k) < \epsilon.$$

Construct an element a^* in $\tilde{M}$ such that $\rho(a_k, a^*) \to 0$ as $k \to \infty$.

3 Compactness

3.1 Let $C([0, 1])$ be the set of bounded continuous functions on $[0, 1]$ with the usual sup norm. So

$$d(f, g) = \sup_{0\le x\le 1} |f(x) - g(x)|$$

Given a function $g \in C([0, 1])$, let $U_g = \{f : f(x) > g(x) \quad \forall x \in [0, 1]\}$. Prove that U is open.

3.2 Prove or disprove the following:

(a) A is finite and U is a open subset of $\mathbb{R}$. If $A \subseteq U$, there exists an $\epsilon > 0$ such that for all $x \in A$, $N(x, \epsilon) \subseteq U$.

(b) P is countable and U is a open subset of $\mathbb{R}$. If $P \subseteq U$, there exists an $\epsilon > 0$ such that for all $x \in P$, $N(x, \epsilon) \subseteq U$.

(c) F is closed and U is a open subset of $\mathbb{R}$. If $F \subseteq U$, there exists an $\epsilon > 0$ such that for all $x \in F$, $N(x, \epsilon) \subseteq U$.

(d) K is compact and U is a open subset of $\mathbb{R}$. If $K \subseteq U$, there exists an $\epsilon > 0$ such that for all $x \in K$, $N(x, \epsilon) \subseteq U$.

3.3 This is a continuation of problem 2.1. For $x, y \in \mathbb{R}$, let

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}$$

We already showed that d is a metric on $\mathbb{R}$.

(a) Prove that with this metric, the entire space of $\mathbb{R}$ is not compact even though it is closed and bounded.

(b) Can you characterize all the compact sets in $(\mathbb{R}, d)$?

3.4 A metric space X is said to be *totally bounded* if for any $\epsilon > 0$ it can be covered by finitely many ϵ-balls. Also, a metric space is complete, if every Cauchy sequence in X converges. Prove that X is compact if and only if X is complete and totally bounded.

3.5 Define the "distance" between two subsets of $\mathbb{R}$ by $\rho(A, B) = \inf_{x \in A, y \in B} |x - y|$.

(a) Is ρ a metric on the power set $2^{\mathbb{R}}$ (i.e the collection of all subsets of $\mathbb{R}$)?

(b) If A is closed and B is compact, show that $\rho(A, B) = 0$ if and only if $A \cap B$ is nonempty.

(c) If A and B are closed, does it follow that $\rho(A, B) = 0 \implies A \cap B \neq \emptyset$?

4 Calculus

4.1 Prove the inequality $e^x \geq 1 + x$ for all $x \in \mathbb{R}$.

4.2 $f : \mathbb{R}^k \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^n$ are differentiable. Prove the chain rule for the composition $g \circ f$.

4.3 A C^2 function $f : \mathbb{R} \to \mathbb{R}$ is convex if $f''(x) > 0$ for all x. Show that, for any finite collection of points $a_1, a_2, \ldots, a_n$, we have the inequality

$$\frac{f(a_1) + f(a_2) + \cdots + f(a_n)}{n} \geq f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right).$$

(Hint: Draw a picture. Start with $n = 2$)

4.4 (a) Give an example of a function of two variables that is discontinuous at the origin, but whose partial derivatives at the origin exist.

(b) Give an example of a function of two variables all of whose directional derivatives exist at the origin, but the function itself is not differentiable at the origin.

4.5 An open subset $O \subset \mathbb{R}^n$ has the property that for any pair of points $x, y \in O$, there is a differentiable function $\gamma : [0, 1] \to O$ such that $\gamma(0) = x$ and $gamma(1) = y$. $f : O \to \mathbb{R}^m$ is differentiable and $Df = 0$ on O. Show that f is a constant on O.

(The Hypothesis can be slightly weakened. It is sufficient to assume that O is connected and open.)

4.6 Evaluate the derivatives of the following matrix functions:

(a) $inv : GL(n) \to GL(n)$ given by $inv(M) = M^{-1}$.

(b) The determinant function which maps $GL(n)$ to $\mathbb{R}$.

4.7 Show that every point p on the sphere $x^2 + y^2 + z^2 = 1$ has a (3 dimensional) neighborhood U such that there is a smooth, one to one mapping of an open neighborhood V of the origin in $\mathbb{R}^3$ such that the plane $z = 0$ maps to the surface of the sphere.

4.8 Let f and $\frac{\partial f}{\partial y}$ be continuous on $[0,1] \times [0,1]$ and assume that $p, q : [0,1] \to [0,1]$ are differentiable. Define

$$F(y) = \int_{p(y)}^{q(y)} f(x,y)dx, \quad y \in [0,1]$$

Use the chain rule to find $F'(y)$. Hint: Consider $G(x_1, x_2, x_3) = \int_{x_1}^{x_2} f(t, x_3)dt$.

4.9 Let $f(x,y) = (x^2 - y^2)/(x^2 + y^2)^2$. Show that

$$\int_0^1 \left(\int_0^1 f(x,y)dx \right) dy = -\pi/4, \quad but \quad \int_0^1 \left(\int_0^1 f(x,y)dy \right) dx = \pi/4$$

4.10 Show that in a neighborhood of the origin, the systems of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

can be solved for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x. What can you say (using the implicit function theorem) about solving them for x, y, z in terms of u?

4.11 Let F denote the vector field $(x^2 - z^2, 2xy, z)$ on $\mathbb{R}^3$. Compute in two different ways the surface integral

$$\int \int_T F \cdot n \, dA$$

where T denotes the surface of the tetrahedron bounded by $x \geq 0, y \geq 0, z \geq 0, x+y+z \leq 1$ and n denotes the outward normal to T. Use the two answers to verify the divergence theorem.

4.12 Compute in two different ways the line integral

$$\oint_C ydx - xdy + z^2dz$$

where C is the intersection of the paraboloid $z = x^2 + 4y^2$ with the cylinder $x^2 + y^2 = 9$, traversed counter-clockwise when viewed from the point $(0, 0, 100)$. Use the two answers to verify Stokes' theorem

4.13 A smooth function F on the (open) unit disc in $\mathbb{R}^2$ satisfies $F(0,0) = (0,0)$ and $\|F(x)\| \geq 3\|x\|$.

(a) Show that F is one to one in a neighborhood of the origin.

(b) Let $F = (u, v)$ in components. Show that, for all $a < 1$, the integral

$$\frac{1}{2\pi} \oint_{\|r\|=a} \frac{u \nabla v - v \nabla u}{u^2 + v^2} \cdot dr$$

is an integer and this value does not depend on a.

(c) Can you determine the value of the integral in terms of the Jacobian $A = DF(0,0)$ of the mapping F at the origin?

(d) Can you construct a smooth function F on the (open) unit disc in $\mathbb{R}^2$ satisfying $F(0,0) = (0,0)$, $F(x) \neq (0,0)$ for $x \neq (0,0)$, such that the value of the integral is 2?

4.14 $M : \mathbb{R}^2 \to \mathbb{R}^2$ is the mapping

$$M(x, y) = (2x - \sin(xy), \frac{1}{3}y + x^2 - y^2).$$

(a) Show that M is invertible in a neighborhood of the origin.

(b) The *unstable manifold* of the origin is the set $\Gamma_u = \{(x, y) | M^{-n}(x, y) \to (0, 0) \text{ as } n \to \infty\}$. Show that Γ_u is invariant under M.

(c) Show that, there is a unique function f such that $f(x) \to 0$ when $x \to 0$, and the unstable manifold is given by the graph of the function f for sufficiently small x, i.e.

$$\Gamma_u \cap B_\epsilon(0, 0) = \{(x, f(x)) | |x| < \epsilon\} \cap B_\epsilon(0, 0)$$

5 Complex analysis

5.1 Let $f(z)$ be analytic at c. Write $f'(c) = re^{i\theta}$. Write $z = x+iy$ and $f(z) = u(x, y)+iv(x, y)$. We can think of f as a map $(u(x, y), v(x, y)$ from $\mathbb{R}^2 \to \mathbb{R}^2$. The total derivative of this map at c is a two by two matrix. Find it in terms of r and θ. Express the direction derivatives of the map on $\mathbb{R}^2$ in terms of r and θ.

5.2 Define

$$f(z) = \sum_{k=1}^{\infty} a_k (z - z_0)^k$$

(a) If the radius of convergence of the above power series is r, then show that the radii of convergence for the series obtained by differentiating and integrating the above series termwise is also r.

(b) Show that the termwise differentiated series converges uniformly on every disk of the form $|z - z_0| \leq \rho < r$. Use this to show that $f(z)$ is given by termwise differentiating the above series, for any compact subset of the disk $|z - z_0| < r$.

5.3 Suppose that γ is a piecewise smooth positively, counterclockwise oriented, simple closed curve. Use Green's Theorem to show that the value of the integral

$$\oint_\gamma \frac{dz}{z - p}$$

equals 0 if p is outside γ and $2\pi i$ if p is inside γ.

5.4 Let $f(z)$ be analytic at z_0. Prove that for sufficiently small ϵ,

$$\frac{1}{2\pi i}\oint_{|z-z_0|=\epsilon}\frac{f(z)}{(z-z_0)^n}\,dz = \frac{f^{(n-1)}(c)}{(n-1)!}$$

The contour is the circle centered at z_0 with radius ϵ traversed in the counterclockwise direction. This is a standard theorem in complex analysis books. The exercise is to prove it directly from the statement of Cauchy's theorem given in the notes. Hint: power series.

5.5 If $f(z)$ is analytic on the closed disk $B(z_0, r)$, show that there is a constant C such that the derivatives of f at z_0 can be bounded by

$$\left|f^{(n)}(z_0)\right| \le \frac{Cn!}{r^{n+1}}$$

5.6 For each of the following functions f and z_0, determine if the function has a pole or essential singularity at z_0. In the case of a pole determine the order of the pole and the principal part.

(a) $f(z) = \frac{1}{z\sin(z)}$, $z_0 = 0$

(b) $f(z) = \frac{1}{(z^2+1)^2}$, $z_0 = i$

(c) $f(z) = \exp(-1/z)$, $z_0 = 0$.

(d) $f(z) = \tan^2(z)$, $z_0 = \pi/2$.

5.7 A Möbius transformation , or a fractional linear transformation, is a mapping of the form

$$w = f(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$. This can be extended to the Riemann sphere $= \mathbb{C} \cup \{\infty\}$

$$f(\infty) = a/c, \quad f(-d/c) = \infty$$

(a) Show that the Mobius transformations form a group.

(b) Show that the Mobius transformations map circles to circles on the Riemann sphere (Note: A straight line on $\mathbb{C}$ is considered a circle through ∞ on the Riemann sphere).