INTEGRATION WORKSHOP 2004
PROJECT ON LINEAR DIFFERENTIAL EQUATIONS AND THE JORDAN FORM

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For a complex number λ and a positive integer d, let $Q(\lambda, d)$ be the space of functions $p(x)e^{\lambda x}$ where p is a polynomial of degree $< d$. Let $T : Q(\lambda, d) \to Q(\lambda, d)$ be differentiation with respect to x. Find a basis of $Q(\lambda, d)$ with respect to which T has the Jordan form.

Let Q be any finite dimensional space of differentiable functions of x satisfying $f \in Q \Rightarrow \frac{df}{dx} \in Q$. Prove that there exist complex numbers $\lambda_1, \ldots, \lambda_k$ and integers $d_1, \ldots, d_k$ such that

$$Q = \bigoplus_{j=1}^{k} Q(\lambda_j, d_j).$$

Hint: Consider the Jordan form of $\frac{d}{dx}$ on Q.

Let y be a function of x satisfying the differential equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0$$

where $a_j \in \mathbb{C}$ for $j = 1, \ldots, n$. Apply the previous parts to the space spanned by y and its derivatives to prove that $y = \sum_j p_j(x) e^{\lambda_j x}$ for suitable complex numbers λ_j and polynomials p_j. What can you say about the λ_j and the degrees of the p_j in terms of the coefficients a_j?

Given time, connect this point of view with that of a system of n first order differential equations with constant coefficients and its solution via the matrix exponential.