Combinatorial Topology of Dynamical Systems

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August 5, 2004

Solving a system of differential equations such as

$$\begin{aligned} \dot{x} &= f_1(x,y) \\ \dot{y} &= f_2(x,y) \end{aligned} \tag{1}$$

by finding a solution in terms of elementary functions is usually impossible. More often than not, the analysis of such systems turns towards questions about the the "overall behavior" of the system. Where are the *fixed points*? (i.e., points (x_0, y_0) such that $f_1(x_0, y_0) = f_2(x_0, y_0) = 0$.) Do large sets of initial conditions contract or expand? Do the equations describe motion in the plane or something more complicated?

For example, consider the motion of a small water droplet on the surface of a spherical glass ball determined by the effects of gravity and friction. After placing the water droplet on the ball, the drop will begin to move along the surface towards the lowest point of the sphere due to the influence of gravity. (Surface tension and other effects keep the droplet attached to the ball if the drop is small enough.) As the droplet approaches the bottom of the ball, gravity tries to pull the droplet off of the ball rather than along the surface and the surface tension of the droplet resists this causing its speed to slow and eventually come to rest at the base of the ball. In theory, one could place the water droplet on the very top of the sphere and it would not move at all.

Here, the motion of the droplet from an initial point on the surface of the ball is described by a curve on a 2-dimensional sphere. The collection of all possible trajectories of this system is called the *flow* of the system. Notice that this system has two fixed points. The obvious one is at the south pole S of the sphere and the other is at the north pole N of the sphere. In this system, a droplet always falls away from N and towards S.

1. Try to sketch the trajectories of a flow on the sphere with 4 fixed points. Try to sketch the trajectories of a flow different from the water droplet example with only two fixed points. Try to draw the trajectories of a flow on the sphere with only one fixed point. Do the same for an ellipsoid.

Our goal in this project is to relate the numbers and types of fixed points of a system to the topology of the surface to which it is constrained.

1 Vector Fields on the Plane and Flows

The planar system (1) of differential equations gives rise to a map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ via $F(x, y) = (f_1(x, y), f_2(x, y))$ which can be used to analyze the flow of the sytem. (Recall: The flow $\Phi_t(x, y)$ is describes the trajectory of the intial data (x, y) at time t subject to the system (1), i.e., $\frac{d}{dt}\Phi_t = (f_1, f_2)$ for all $(x, y) \in \mathbb{R}^2$.)

2. Expand F in a Taylor series to show that system is approximated near a fixed point by $\dot{X} = DF_{X_0}$ where $X = (x, y)$ and DF_{X_0} is the Jacobian matrix of F at the fixed point X_0.

Thus, the fundamental properties of the flow near a fixed point X_0 are determined by the eigenvalues and eigenvectors of the Jacobian matrix DF_{X_0}. Let λ_1, λ_2 denote the eigenvalues of DF_{X_0}. The following cases illustrate some of the fundamental types of fixed points.

source The real parts of both λ_1, and λ_2 are positive. E.g., $\begin{aligned}\dot{x} &= 3x + y\\ \dot{y} &= x + 3y\end{aligned}$

saddle The real part of one eigenvalue is positive, the real part of the other is negative. E.g., $\begin{aligned}\dot{x} &= y\\ \dot{y} &= x\end{aligned}$

center The real parts of both eigenvalues are zero. E.g., $\begin{aligned}\dot{x} &= -y\\ \dot{y} &= x\end{aligned}$

sink The real parts of both eigenvalues are both negative. E.g., $\begin{aligned}\dot{x} &= -15x - 5y\\ \dot{y} &= -5y - 15x\end{aligned}$

3. Show that if the fixed point X_0 is *non-degenerate*, i.e., $\det DF_{X_0} \neq 0$, then these are the only possibilities.

4. Find and classify the fixed points of the following vector fields, and plot their phase portraits.

   (a) $F(x, y) = (y + y^2, -x + xy)$
   (b) $F(x, y) = (x(3 - x - 2y), y(2 - x - y))$

5. Let $z = x + iy$, $\bar{z} = x - iy$, and define vector fields on $\mathbb{R}^2$ by

$$(\dot{x}, \dot{y}) = (\mathrm{Re}(z^k), \mathrm{Im}(z^k))$$
$$(\dot{x}, \dot{y}) = (\mathrm{Re}(\bar{z}^k), \mathrm{Im}(\bar{z}^k))$$

   where k is a natural number. Classify the fixed points of these vector fields and sketch phase portraits near the origin for $k = 0, 1, 2$. Qualitatively describe how increasing k changes the vector field near the origin.

6. The locus of points $(x, y, z) \in \mathbb{R}^3$ such that $x^2 + y^2 + z^2 = 1$ is a unit sphere centered at the origin in $\mathbb{R}^3$. Show that the vector field $F(x, y) = (zx, zy, x^2 + y^2)$ defines a tangent vector field on the unit sphere and find the fixed points.

2 Euler Characteristic

A *graph* in $\mathbb{R}^2$ is a finite set of distinct points $\{v_1, v_2, \ldots, v_n\}$ called *vertices* together with a collection of simple paths connecting pairs of (not necessarily distinct) vertices called *edges*. A graph in $\mathbb{R}^2$ is said to be *planar* if no edges intersect. A graph is said to be *connected* if it is a path-connected subset of $\mathbb{R}^2$. Let G be a planar graph in $\mathbb{R}^2$, and denote by V the set of vertices of G. Let E denote the set of edges of G. It is a theorem that $\mathbb{R}^2 \setminus (V \cup E)$ is a union of connected components which we will call *faces*. Let F denote the set of faces of G. Only one of the faces of G is an unbounded subset of $\mathbb{R}^2$, and all other faces are contractible sets (that is, equivalent to a polygon/disk).

7. Let S be a finite set, and denote by $|S|$ the cardinality of S. Prove that $|V|-|E|+|F| = 2$ for any connected planar graph G. (*Hint: Try induction on the number of edges.*)

The two-dimensional sphere as a topological space can be realized as the one-point compactification of $\mathbb{R}^2$. Using this idea we can extend the notion of a non-self-intersecting graph in $\mathbb{R}^2$ to a non-self-intersecting graph on S^2. Note that the one unbounded set now becomes a contractible set on S^2.

8. As another example of a surface T consider the subset of $\mathbb{R}^3$ defined as the locus of points (x, y, z) satisfying $(\sqrt{x^2 + y^2} - 2)^2 + z^2 - 1 = 0$. What does this surface look like? (*Hint: Try drawing level sets or change to cylindrical coordinates.*)

Notice that the complement in $\mathbb{R}^3$ of this surface T consists of two connected components (an inside and an outside), making it possible to define a continuous function from $T \to \mathbb{R}^3$ assigning the outward pointing normal vector to each point of T. A surface for which such a "normal" function is well-defined is said to be *orientable*.

9. Convince yourself that topologically this surface is the same as the topological space obtained by removing two disjoint open disks from the sphere and identifying the pair of boundary circles to the ends of a cylinder.

By an orientable *surface of genus g* we will mean a sphere with g cylinders attached in this manner. Thus, the sphere is a surface of genus 0, and T in the above example is a surface of genus 1.

10. The fact that $|V| - |E| + |F|$ is the same for any connected planar graph extends to the case of a graph on the sphere but does not extend to the torus. Why? However if we define a connected *graph G on a surface* Σ_g of genus g to be a collection of vertices and non intersecting edges such that $\Sigma_g \setminus (V \cup E)$ is a disjoint union of contractible sets then $|V| - |E| + |F|$ is independent of the graph on Σ_g. Prove this.

11. As the sum $|V| - |E| + |F|$ is independent of the graph on Σ_g, we can define a number associated to Σ_g via $\chi(\Sigma_g) = |V| - |E| + |F|$ using any graph on Σ_g. The number $\chi(\Sigma_g)$ is called the Euler Characteristic of the surface. Compute this for Σ_g. (*Hint: Use the construction of Σ_g as the two-dimensional sphere with g cylinders attached.*)

3 The Index of a Fixed Point

Above, we studied the stability of fixed points. Here we define another property of fixed points, which tells us "how the vector field winds about the fixed point".

First, we consider vector fields in the plane $\mathbb{R}^2$ and how they change along C^1 *simple closed curves.* A C^1 curve is a C^1 function $c(t) : [0, 1] \to \mathbb{R}^2$, and a curve is closed if $c(0) = c(1)$ and simple if it is otherwise not self-intersecting. We will call a C^1 simple closed curve a C^1_{sc} curve.

Given a vector field $F = (p(x, y), q(x, y))$ in $\mathbb{R}^2$ and a C^1_{sc} curve C, we can calculate the total change $\Delta\Theta$ in the angle

$$\Theta = \tan^{-1} \frac{q(x, y)}{p(x, y)}$$

between the vector (p, q) and the x-axis as the point (x, y) transverses C exactly once in the positive direction (the direction of increasing t).

Definition: Given a C^1_{sc} curve C in $\mathbb{R}^2$ and a continuously differentiable vector field F in $\mathbb{R}^2$ that has no fixed points on C, the integer

$$I_F(C) = \frac{\Delta\Theta}{2\pi}$$

is called the *index of C relative to the vector field F.*

12. Show that if $F = (p(x, y), q(x, y))$, then

$$I_F(C) = \frac{1}{2\pi} \int_C \frac{pdq - qdp}{p^2 + q^2}.$$

13. Compute the index of the following vector fields with respect to the unit circle C centered at the origin.

    (a) $F(x, y) = (x, y)$

    (b) $F(x, y) = (-x, -y)$

    (c) $F(x, y) = (-y, x)$ (Note that this vector field is tangent to circles centered at the origin.)

    (d) $F(x, y) = (x, -y)$

14. Prove the following properties of the index of a curve relative to a vector field:

(a) Suppose that the curve C can be continuously deformed into the curve C' without passing through a fixed point. Then, $I_F(C) = I_F(C')$.

(b) If C does not enclose any fixed points, then $I_F(C) = 0$.
(Suggestion: Think about the vector field along curves contained in small disks of radius r as r goes to 0 and use part (a).)

This last exercise allows us to make the following definition:

Definition: Let x_0 be a fixed point of a vector field F. The *index of x_0 with respect to F* is defined to be

$$I_F(x_0) = I_F(C),$$

where C is any positively oriented C^1_{sc} curve such that x_0 is the only fixed point of F contained in the interior of C. By *positively oriented* we mean that the region enclosed by C is on the left-hand-side of someone walking in the direction of increasing t.

15. Is the index $I_F(x_0)$ well defined?

16. Again let $z = x + iy$, $\bar{z} = x - iy$, and define vector fields on $\mathbb{R}^2$ by

$$(\dot{x}, \dot{y}) = (\mathrm{Re}(z^k), \mathrm{Im}(z^k))$$

$$(\dot{x}, \dot{y}) = (\mathrm{Re}(\bar{z}^k), \mathrm{Im}(\bar{z}^k))$$

where k is an integer. Compute the index of the fixed point at the origin.

17. Show that if a closed curve C encloses a finite number of fixed points $x_1, \ldots x_n$, then

$$I_F(C) = \sum_{j=1}^{n} I_F(x_j).$$

(Suggestion: Recall the proof of a similar theorem in complex analysis!)

So far we have dealt with indices with respect to vector fields in the plane. How can we define the index of a fixed point with respect to a vector field tangent to a surface?

18. Consider again the unit sphere in $\mathbb{R}^3$ with the tangent vector field $F = (zx, zy, x^2 + y^2)$ and fixed points at the north N and south S poles. Realizing the sphere in a neighborhood of S as the graph of a function $z = f(x, y)$, define the projection of the vector field near S to the plane and compute the index of the fixed point of the projected vector field. (Suggestion: Compute the index of the vector field relative to a small circle about the origin.)

By similarly mapping a vector field on any surface to the plane and computing the index of the projected vector field, we can define the index of a fixed point for a vector field on a surface. But, notice that the orientation of the surface becomes an issue.

19. On an orientable surface Σ an orientation is determined by a normal vector field as described above. Give a definition of a *positively oriented curve* on Σ.

One must check that the index is independent of the chosen projection and that the properties of indices for vector fields in $\mathbb{R}^2$ also hold for indices relative to vector fields on surfaces. Then we can make the following two definitions:

Definition: Let x_0 be a fixed point of a vector field F on an oriented surface Σ. The *index of x_0 with respect to F* is defined to be

$$I_F(x_0) = I_F(C),$$

where C is any positively oriented C^1_{sc} curve such that x_0 is the only fixed point of F contained in the interior of C. The *interior* of C is the region to the left of someone walking in the positive direction on the curve.

Definition: The *index of a surface S with respect to the vector field F on S* is defined to be the sum

$$I_F(S) = \sum_{j=1}^{n} I_F(x_j)$$

of the indices of the critical points x_j of F on S.

A priori, this definition depends on both the vector field F and the surface S. The Poincaré-Hopf Theorem says that the index is actually independent of F!

4 The Poincaré-Hopf Index Theorem

Theorem: The sum $I_F(S)$ of the indices of a vector field F on a compact, connected, orientable surface S is equal to the Euler characteristic of S.

Our strategy is to first prove the result for the sphere and then to use the realization of Σ_g as a sphere with g cylinders attached to extend the result to compact, connected, orientable surfaces of arbitrary genus.

Let F be a vector field on a sphere.

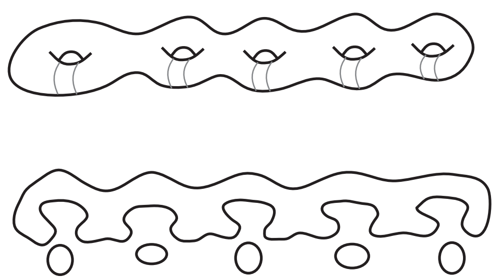
20. Let p be a point on the sphere that is not a fixed point for F. Argue that exists a C^1_{sc} curve C enclosing p and no fixed points of F such on C the field F is essentially constant.

21. The curve C divides the sphere into two connected components. Argue that sphere minus the component A containing p is "the same" as a disk. We will denote this space by $S^2 \setminus A$.

22. Our task is to show that the index of the curve C relative to F in $S^2 \setminus A$ is 2. The ideas needed to make this computation completely rigorous are the heart and soul of the first year core-course Geometry/Topology, so we can only give a hands-on approach here. Using a pen, draw a vector field on a tennis ball (perhaps one of those whose flow you sketched in problem 1). Find a fixed-point free region on your tennis ball and draw a small circle. Turn the ball over, and track how many times the vector field winds around as you traverse the curve.

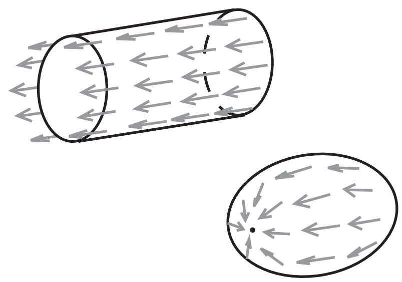
This is very simliar to an old bar trick. The one where you take two quarters and place them flat on the surface of a table so that they're tangent at one point and then roll one quarter around the boundary of the other keeping the other fixed. The moving quarter revolves twice as it completes one trip around the fixed quarter.

Thus, (with some faith) we've established that the index of the sphere is 2. To prove the theorem, we'll use the presentation of the general surface of genus g as a sphere with g cyclinders attached and perform "surgery" to reduce to the case of the sphere. The figure shows a surface of genus 5. The five cylinders are indicated. Removing the cylinders and contracting their boundary circles to points produces 6 disjoint spheres.



24. Convince yourself that we can assume without loss of generality that the vector field on the surface has no fixed points in each of the cylinders.

25. Argue that contracting the boundary circles of one of the cylinders to points and surgically excising the resulting sphere from the new surface introduces four fixed points, a source and a sink on the resulting sphere, and a corresponding sink and source (respectively) on the surface. The figure may be helpful.



26. Compute the index of the surgically altered surface in two ways. First, count the index of the original surface plus the contributions from surgery. Second, realize that the surgically altered surface is a collection of $g+1$ spheres, each of which has index 2. Finish the “proof” of the theorem.