Exploring the Exotic Setting for Algebraic Geometry

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1 Introduction

In this project, we will describe the basic topology used in algebraic geometry, called the Zariski topology (named after Oscar Zariski). This project will require some familiarity with basic commutative algebra. Some definitions and a couple of useful facts are given in an appendix.

In some sense, algebraic geometry is a generalization of linear algebra. In linear algebra, one studies simultaneous solutions to systems of linear equations. Algebraic geometry is based on studying simultaneous solutions to systems of polynomial equations. When the equations have degree one in each variable, algebraic geometry reduces to linear algebra. The solution sets are planes in a vector space and are "flat" in a naïve sense. When the polynomials have a higher degree, the geometry is much more rich.

The first exercise gives a couple of simple examples.

Exercise 1.1. Describe the solution set in $\mathbb{R}^3$ to the following systems of equations:

1.

$$\begin{aligned} F_1 = x^2 + y^2 + z^2 &= 1 \\ F_2 = x + y &= 0 \end{aligned}$$

2.

$$\begin{aligned} F_1 = x^2 + y^2 - z^2 &= 1 \\ F_2 = x &= 0 \end{aligned}$$

In these examples, the topology is natural because the solution sets are subsets of $\mathbb{R}^3$. However, what if we are considering polynomials over a field such as $\overline{\mathbb{F}_p}$ or $\mathbb{Q}_p$ (the field of p-adic numbers)?

In this project, we will define a topology on solution sets to polynomial equations with coefficients in any field. The topology will be directly related to the algebra of the polynomial ring. In fact, the definition will work for any commutative ring with 1, not just a polynomial ring over a field. In this way, tools in algebraic geometry are useful in areas such as algebraic number theory.

2 $\mathrm{Spec}(R)$ as a Set

First we will translate solution sets of polynomials into ring-theoretic language. This will lead to a definition of a set called $\mathrm{Spec}(R)$ (where R is a commutative ring with 1). In the next section we will put a topology on $\mathrm{Spec}(R)$, and then we will conclude by exploring features of this topology, some of which are very unusual.

As a convention, all of our rings are commutative with 1, and $0 \neq 1$. (There has been some work on generalizing these definitions to the zero ring.)

To motivate the definitions that follow, we again suppose we have a simultaneous solution set to a system of polynomial equations. Let the polynomials be denoted $F_1, \ldots, F_m$. Let $V(\{F_1, \ldots, F_m\})$ be this solution set.

For our motivation, these polynomials have coefficients in an algebraically closed field $k = \overline{k}$ and involve n indeterminants $x_1, \ldots, x_n$. Then for each $i = 1, \ldots, m$, $F_i \in k[x_1, \ldots, x_n]$. Then $V \subset k^n$. Let $(F_1, \ldots, F_m) \subset k[x_1, \ldots, x_n]$ be the ideal generated by $F_1, \ldots, F_m$.

Exercise 2.1. Show that if $f \in (F_1, \ldots, F_m)$, then $f \in k[x_1, \ldots, x_n]$ satisfies $f(p) = 0$ for every $p \in V(\{F_1, \ldots, F_m\})$.

Exercise 2.2. Show that the converse is false by considering the one polynomial $F = x^2$ in $\mathbb{C}[x]$.

Exercise 2.3. Let $V \subset k^n$ be any subset. Show that

$$\{f \in k[x_1, \dots, x_n] \mid f(p) = 0 \ \forall p \in V\}$$

is an ideal in $k[x_1, \dots, x_n]$.

Given a subset V, the ideal defined in Exercise 2.3 is denoted $I(V)$.

Given an ideal $J \subset k[x_1, \dots, x_n]$, define $V(J)$ to be the set

$$V(J) := \{p \in k^n \mid f(p) = 0 \ \forall \ f \in J\}.$$

Definition 2.4. A subset $V \subset k^n$ is an **algebraic set** if $V = V(J)$ for some ideal $J \subset k[x_1, \dots, x_n]$.

Exercise 2.5. Let $J = (F_1, \dots, F_m) \subset k[x_1, \dots, x_n]$. Show that $V(J) = V(\{F_1, \dots, F_m\})$.

Exercise 2.6. Show that $J \subset I(V(J))$, but this inclusion may be proper.

We would like to have a perfect dictionary between algebraic sets $V \subset k^n$ and ideals $J \subset k[x_1, \dots, x_n]$ given by $J \leftrightarrow V(J)$. However, Exercise 2.2 shows that this fails: $V(x) = V(x^2)$ and in fact $V(x) = V(x^n)$ for any n.

This problem is remedied if we restrict to **radical ideals**.

Definition 2.7. Let $I \subset R$ be an ideal in a ring. The **radical** of I is the set

$$\sqrt{I} := \{f \in R \mid \ \exists l \text{ such that } f^l \in I\}.$$

If $I = \sqrt{I}$, we say that I is a **radical ideal**.

Exercise 2.8. Show that if $I \subset R$ is an ideal, $\sqrt{I}$ is an ideal and $I \subset \sqrt{I}$. Show that $\sqrt{\sqrt{I}} = \sqrt{I}$, so that $\sqrt{I}$ is radical.

Exercise 2.9. Let $R = k[x]$. Show that $\sqrt{(x^n)} = (x)$ for any n and conclude that (x) is a radical ideal while (x^n) is not a radical ideal for $n > 0$.

Exercise 2.10. Let $V \subset k^n$ be any set. Show that $I(V)$ is radical.

There is a perfect dictionary between algebraic sets in k^n and radical ideals in $k[x_1, \ldots, x_n]$. This is a consequence of the following nontrivial theorem called Hilbert's Nullstellensatz.

Theorem 2.11. Let k be an algebraically closed field and $I \subset k[x_1, \ldots, x_n]$ be an ideal. If $f \in k[x_1, \ldots, x_n]$ vanishes at every $p \in V(I)$, then $f^r \in I$ for some integer $r > 0$.

Corollary 2.12. If k is an algebraically closed field, there is a one-to-one inclusion-reversing correspondence between algebraic sets $V \subset k^n$ and radical ideals in $k[x_1, \ldots, x_n]$ given by $V \mapsto I(V)$ and $J \mapsto V(J)$.

Exercise 2.13. Prove Corollary 2.12 using the Nullstellensatz.

Another consequence of the Nullstellensatz helps motivate the definition of $\operatorname{Spec}(R)$.

Corollary 2.14. If k is an algebraically closed field, every maximal ideal $\mathfrak{m} \in k[x_1, \ldots, x_n]$ has the form

$$\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$$

where each $a_i \in k$.

This gives a one-to-one correspondence between points $(a_1, \ldots, a_n) \in k^n$ and maximal ideals $\mathfrak{m} \in k[x_1, \ldots, x_n]$.

Now consider again the system of polynomial equations given by

$$F_1 = 0, \ldots, F_m = 0$$

where $F_i \in k[x_1, \ldots, x_n]$ with $k = \overline{k}$. Let $J = (F_1, \ldots, F_m)$. Then we are considering the algebraic set $V(J)$. This set of points corresponds to a set of maximal ideals in $k[x_1, \ldots, x_n]$. Call this set $\mathcal{M}$ for now.

For any ring R, the set of all maximal ideals of R is denoted $\operatorname{Max}(R)$. This is called the "max-spectrum" of R.

Let $S = k[x_1, \ldots, x_n]$. Then $\mathcal{M} \subset \operatorname{Max}(S)$.

Exercise 2.15. Show that a maximal ideal $\mathfrak{m} \in \mathcal{M}$ if and only if $J \subset \mathfrak{m}$. Use the correspondence theorem to conclude that $\mathcal{M} = \operatorname{Max}(S/J)$.

By Exercise 2.15, we have defined algebraic sets in terms of the intrinsic algebra of a ring. This suggests that we should focus on $\mathrm{Max}(R)$.

The problem with $\mathrm{Max}(R)$ is that it doesn't quite work *functorially*. In addition to our sets being defined in terms of the algebra of rings, we would like the functions between our sets to correspond to ring homomorphisms in a natural way. This doesn't work for maximal ideals.

Exercise 2.16. Give an example of a ring homomorphism $f : R \to S$ and a maximal ideal $\mathfrak{m} \subset S$ such that $f^{-1}(\mathfrak{m})$ is not maximal in R.

The way to solve this is to enlarge our set from maximal ideals to prime ideals.

Exercise 2.17. Show that if $f : R \to S$ is a ring homomorphism and $\mathfrak{p} \subset S$ is a prime ideal, then $f^{-1}(\mathfrak{p})$ is prime.

The right set to consider is the set of all prime ideals of a ring R.

Definition 2.18. Let R be a ring. The **prime spectrum of** R is the set

$$\mathrm{Spec}(R) := \{\mathfrak{p} \subset R \mid \mathfrak{p} \text{ is a prime ideal}\}$$

of prime ideals in R.

Exercise 2.17 shows that a ring homomorphism $f : R \to S$ gives rise to a function $f^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ given by $f^*(\mathfrak{p}) = f^{-1}(\mathfrak{p})$.

Exercise 2.19. Identify the set $\mathrm{Spec}(R)$ for the following rings:

1. $R = \mathbb{Z}$
2. $R = k$, k is any field
3. $R = \mathbb{C}[x]$
4. $R = \mathbb{R}[x]$
5. $R = k^n$
6. $R = \mathbb{Z}/(143)$

3 A Topology on $\text{Spec}(R)$

Given a set of polynomial equations

$$F_1 = 0, \ldots, F_m = 0$$

we defined the ideal $I \subset S = k[x_1, \ldots, x_n]$ generated by the F_i. We defined $V(I)$ to be the points $p \in k^n$ where $f(p) = 0$ for all $f \in I$, which corresponded to

$$\text{Max}(S/I) = \{\mathfrak{m} \in \text{Max}(S) \mid I \subset \mathfrak{m}\}.$$

Since the right set to consider is the set of prime ideals of a ring, we replace maximal ideals by prime ideals and now define

$$V(I) := \{\mathfrak{p} \in \text{Spec}(S) \mid I \subset \mathfrak{p}\}.$$

Note that this definition makes sense for **any** ring S and ideal $I \subset S$. In our case where we are looking at solutions to the polynomial equations, the maximal ideals corresponding to the points in the solution set are contained in $V(I)$, but there are some other points in $V(I)$ as well.

Exercise 3.1. Prove that for any ideal $I \subset R$ in any ring R,

$$V(I) = \text{Spec}(R/I).$$

We will now define a topology on $\text{Spec}(R)$. Define the closed sets of $\text{Spec}(R)$ to be precisely the sets $V(I)$ for some ideal $I \subset R$. We need to show that this is a topology.

Exercise 3.2.

1. Show that for two ideals $I, J \subset S$,

$$V(I) \cup V(J) = V(IJ).$$

2. Show that for any set of ideals $\{I_j\}_{j \in A}$,

$$\bigcap_{j \in A} V(I_j) = V\left(\sum_{j \in A} I_j\right).$$

3. Find ideals I and J in R so that $V(I) = \varnothing$ and $V(J) = \text{Spec}(R)$.

4. Conclude that the set $\{V(I) \mid I \subset R \text{ is an ideal}\}$ defines the closed sets of a topology on $\mathrm{Spec}(R)$.

We can describe a basis for the open sets in the Zariski topology. Let $f \in R$. Then the set

$$D(f) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid f \notin \mathfrak{p}\} = \mathrm{Spec}(R) \setminus V(f)$$

is an open set. We call such sets the **basic open sets of** $\mathrm{Spec}(R)$ because of the following exercise.

Exercise 3.3. Show that the basic open sets of $\mathrm{Spec}(R)$ form a basis for the Zariski topology.

Let us consider again some examples.

Exercise 3.4. Identify the topology on $\mathrm{Spec}(R)$ for the following rings:

1. $R = \mathbb{Z}$
2. $R = k$, k is any field
3. $R = \mathbb{C}[x]$
4. $R = \mathbb{R}[x]$
5. $R = k^n$

We should also determine what functions between $\mathrm{Spec}(R)$ and $\mathrm{Spec}(S)$ we want to consider. They should at least be continuous. If you are familiar with category theory, we are identifying the morphisms in a category (called the category of affine schemes).

Exercise 3.5. Recall that if $f : R \to S$ is a ring homomorphism, we get a function of sets $f^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ given by $f^*(\mathfrak{p}) = f^{-1}(\mathfrak{p})$. Show that this function is continuous. In general, a function $g : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is called a **morphism** if $g = f^*$ for some ring homomorphism $f : R \to S$.

We can redefine what algebraic geometry is in terms of prime spectra.

Definition 3.6. An **affine algebraic variety** is $\mathrm{Spec}(S/I)$ where $S = k[x_1, \ldots, x_n]$, k is algebraically closed and $I = \sqrt{I}$.

Algebraic geometry is the study of algebraic varieties and morphisms between algebraic varieties. The affine algebraic varieties are the building blocks for algebraic varieties. Some authors require other conditions on $\mathrm{Spec}(S/I)$ (such as irreducibility or separability, discussed further below). As usual, you should be aware of any author's conventions when reading their work.

Note that a system of polynomial equations in finitely many variables over an algebraically closed field defines an ideal whose radical gives us an affine algebraic variety. Similarly, given an affine algebraic variety we can consider the system of polynomial equations given by generators of the ideal. There are finitely many because the ring is Noetherian (which will be discussed a little more below).

4 Exploring the Zariski Topology

In this section we will explore some of the more exotic features of the Zariski topology. For example, there are single points that are not closed. The Zariski topology is not Hausdorff, but there is a condition (called separability) which replaces the Hausdorff condition. In some circumstances, every pair of open sets has a nonempty intersection and every open subset is dense. The Zariski topology on products is not the product topology. These strange properties arise because we are forced to use prime ideals instead of maximal ideals in our definition of $\mathrm{Spec}(R)$.

In addition to some of the more exotic features of the Zariski topology, we will also look at connectedness and compactness considerations.

Exercise 4.1. Show that a singleton $\{\mathfrak{p}\} \subset \mathrm{Spec}(R)$ is closed if and only if $\mathfrak{p} \in \mathrm{Max}(R)$ (these points $\mathfrak{p}$ are called **closed points**). In particular, if R is not a field, then there are singleton sets in $\mathrm{Spec}(R)$ that are not closed.

Exercise 4.2. Describe the closed points of $\mathrm{Spec}(R)$ for the following rings:

1. $R = \mathbb{Z}$
2. $R = k$

3. $R = \mathbb{C}[x]$

4. $R = \mathbb{R}[x]$

Exercise 4.3. Identify a point $\{\mathfrak{p}\}$ that is dense in $\operatorname{Spec}(R)$ for the following rings:

1. $R = \mathbb{Z}$

2. $R = k[x]$

Such a point is called a **generic point**.

Exercise 4.4. Show that if a topological space X is Hausdorff, $\{p\} \subset X$ is closed for every $p \in X$. Conclude that the Zariski topology is not Hausdorff.

Exercise 4.5. A topological space X is called **irreducible** if any time $X = X_1 \cup X_2$ where $X_1, X_2 \subset X$ are closed, then either $X_1 = X$ or $X_2 = X$. Note that this notion is useless outside of algebraic geometry. Find a necessary and sufficient condition on an ideal $I \subset R$ for $\operatorname{Spec}(R/I)$ to be irreducible.

Exercise 4.6. For a topological space X, prove that the following are equivalent:

1. X is irreducible.

2. Any two nonempty open subsets $U, V \subset X$ have a nonempty intersection.

3. Any nonempty open subset U is dense in X.

Conclude that an irreducible topological space is not Hausdorff. In particular, an irreducible affine algebraic variety is not Hausdorff.

Since the ring $S = k[x_1, \ldots, x_n]$ is of such importance, we give special notation to $\operatorname{Spec}(S)$. Define

$$\mathbb{A}^n_k := k[x_1, \ldots, x_n].$$

This is called **affine n-space**. If the field k is understood it is often omitted from the notation.

Exercise 4.7. Show that the closed points of $\mathbb{A}^2$ are the same as the closed points of $\mathbb{A}^1 \times \mathbb{A}^1$, but the topology on $\mathbb{A}^2$ is not the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$ (even for the subspace topology on the closed points).

Remark 4.8. If you are familiar with the tensor product, the product $\operatorname{Spec}(R) \times \operatorname{Spec}(S)$ is defined to be $\operatorname{Spec}(R \otimes S)$, which does not carry the product topology but does satisfy the appropriate universal property. Note that I did not specify what we are tensoring R and S over. If you like, think of it as $R \otimes_{\mathbb{Z}} S$, but in other circumstances you will use something besides $\mathbb{Z}$. This goes to categorical considerations that I will not elaborate on here.

We can define a condition on $\operatorname{Spec}(R)$ that plays a role in algebraic geometry analogous to that played by the Hausdorff condition in analysis.

Exercise 4.9. Show that a topological space X is Hausdorff if and only if the diagonal

$$\Delta = \{(p, q) \in X \times X \mid p = q\}$$

is closed in the product topology on $X \times X$.

Definition 4.10. The topological space $\operatorname{Spec}(R)$ is **separated** if the diagonal

$$\Delta \subset \operatorname{Spec}(R) \times \operatorname{Spec}(R)$$

is closed.

Remark 4.11. Do not confuse this with a separable topological space!

Exercise 4.12. Show that $\mathbb{A}^n$ is separated (for simplicity you may consider only closed points if you like).

Exercise 4.13. For those familiar with tensor products and universal properties, explain precisely what the diagonal Δ should be in the definition of a separated Zariski topology.

Recall that if S_1 and S_2 are rings, the product ring $S_1 \times S_2$ is the Cartesian product with addition and multiplication defined component-wise, and identity $(1_{S_1}, 1_{S_2})$.

Exercise 4.14. Show that $\operatorname{Spec}(R)$ is disconnected if and only if there are rings S_1, S_2 such that $R = S_1 \times S_2$. Note that this explains why $\operatorname{Spec}(S_1) \times \operatorname{Spec}(S_2) \neq \operatorname{Spec}(S_1 \times S_2)$.

Finally, let us consider compactness. To keep things simple, we will only consider Noetherian rings. Let us recall the definition.

Exercise 4.15. Let R be a ring. Prove that the following are equivalent (you will need Zorn's Lemma):

1. Every ideal $I \subset R$ is finitely generated.
2. Every non-empty set of ideals has a maximal element (with respect to a partial ordering by inclusion).
3. The ring R satisfies the ascending chain condition: If

   $$I_1 \subset I_2 \subset \cdots \subset I_n \subset I_{n+1} \subset \cdots$$

   then there is k such that

   $$I_k = I_{k+1} = \cdots = I_l = \cdots$$

   for all $l > k$.

Such a ring is called a **Noetherian** ring.

Exercise 4.16. Prove that a field is Noetherian. More generally, prove that every PID is Noetherian.

A theorem, called the Hilbert basis theorem, states that if R is Noetherian, so is $R[x]$. By induction, $k[x_1, \ldots, x_n]$ is Noetherian.

Exercise 4.17. Prove that if R is Noetherian and $I \subset R$ is an ideal, R/I is Noetherian.

In algebraic geometry, where we consider rings of the form $k[x_1, \ldots, x_n]/I$, one can see that being Noetherian is a mild assumption.

Exercise 4.18. Prove that if R is Noetherian, $\operatorname{Spec}(R)$ is compact.

Note that in algebraic geometry, we use the term **quasi-compact** instead of compact. This is because the term compact is typically used when the topological spaces are Hausdorff.

5 Concluding Remarks

In your core course on geometry and topology, you will study topological spaces called manifolds. Manifolds are topological spaces constructed by "patching together" open subsets of $\mathbb{R}^n$.

In algebraic geometry, we study algebraic varieties which, analogous to manifolds, are constructed by "patching together" affine algebraic varieties $\mathrm{Spec}(R)$, where the rings may vary! The key difference is that the role of analysis in differential geometry is played by algebra for algebraic geometry.

If one constructs manifolds using $\mathbb{C}^n$ instead of $\mathbb{R}^n$, one can define complex manifolds. Since $\mathbb{C}$ is algebraically closed, the Nullstellensatz holds over $\mathbb{C}$ and it is therefore a natural field to use in algebraic geometry. It is a natural question to compare results for algebraic varieties over $\mathbb{C}$ to complex manifolds. This is the subject of Serre's celebrated GAGA theorem. Here, GAGA stands for the French translation of "Algebraic Geometry Analytic Geometry."

6 Appendix: Commutative Algebra Basics

This appendix is merely to remind the reader of the basic definitions and facts from commutative algebra that are useful in this project. It contains no proofs, no examples, and only a couple of very trivial exercises.

Definition 6.1. A **commutative ring with identity** R is set R together with two binary operations $+$ and $\times$ such that

1. $(R, +)$ is an abelian group with identity $0 \in R$,
2. $\times$ is associative: $(a \times b) \times c = a \times (b \times c)$ for every $a, b, c \in R$,
3. $\times$ is commutative: $a \times b = b \times a$ for all $a, b \in R$,
4. $+$ and $\times$ are distributive: $a \times (b + c) = (a \times b) + (a \times c)$ for all $a, b, c \in R$, and
5. there is an element $1 \in R$ such that $0 \neq 1$ and $a \times 1 = a$ for every $a \in R$.

The result of the binary operation $a \times b$ is usually just written ab.

Definition 6.2. A **ring homomorphism** is a function $f : R \to S$ of sets from the ring R to the ring S such that

1. $f(a + b) = f(a) + f(b)$ for all $a, b \in R$,
2. $f(ab) = f(a)f(b)$ for all $a, b \in R$,
3. $f(0) = 0$, and
4. $f(1) = 1$.

Definition 6.3. An **ideal** $I \subset R$ is a subset of R such that

1. $(I, +)$ is a subgroup of $(R, +)$, and
2. $rI \subset I$ for every $r \in R$.

Definition 6.4. Let $S = \{s_a\}_{a \in A}$ be an arbitrary subset of R. The **ideal generated by** S, denoted (S), is the set

$$(S) = \left\{ \sum_{a \in S} r_a s_a \;\middle|\; r_a \in R,\ r_a = 0 \ \text{ for all but finitely many } \ a \in A \right\}.$$

Exercise 6.5. Prove that (S) is an ideal.

Definition 6.6. If $I \subset R$ is an ideal and I can be written as $I = (S)$ where S is finite, we say that I is **finitely generated**.

Definition 6.7. If $I, J \subset R$ are ideals, then the product IJ is the set of all finite sums of elements of the form ab with $a \in I, b \in J$.

Exercise 6.8. Prove that IJ is an ideal.

Definition 6.9. If $\{I_a\}_{a\in A}$ is a set of ideals, then

$$\sum_{a\in A} I_a = \left(\bigcup_{a\in A} I_a\right).$$

Definition 6.10. If $I \subset R$ is an ideal then the **quotient** R/I is the set of cosets $\{r + I \mid r \in R\}$ endowed with binary operations

$$\begin{aligned}(r+I)+(s+I) &= (r+s)+I\\ (r+I)(s+I) &= rs+I.\end{aligned}$$

Remark 6.11. Since I is an ideal, the binary operations are well defined on cosets and give R/I the structure of a commutative ring with additive identity $0 + I$ and multiplicative identity $1 + I$.

Definition 6.12. A commutative ring with identity R is an **integral domain** if every time $r, s \in R$ with $rs = 0$, either $r = 0$ or $s = 0$.

Definition 6.13. An element $r \in R$ is a **unit** if there is some $s \in R$ such that $rs = 1$.

Exercise 6.14. Prove that if $r \in R$ is a unit, then $(r) = R$.

Definition 6.15. An nonzero element $r \in R$ that is not a unit is **irreducible** if whenever $r = ab$ for $a, b \in R$, either a or b is a unit in R.

Definition 6.16. A nonzero element $r \in R$ is prime if r is not a unit and whenever $p \big| ab$ in R then either $p \big| a$ or $p \big| b$.

Proposition 6.17. If R is an integral domain, then a prime element is always irreducible.

Definition 6.18. A **unique factorization domain** (UFD) is an integral domain R such that for every nonzero element $r \in R$ that is not a unit, we can write $r = p_1 p_2 \cdots p_n$ where p_i is an irreducible in R and the decomposition is unique up to multiplication by units.

Proposition 6.19. If R is a UFD, an element $r \in R$ is irreducible if and only if r is prime.

Definition 6.20. An ideal $I \subset R$ is a **principal** ideal if there is $f \in R$ such that $I = (f)$.

Definition 6.21. A **Principal Ideal Domain** (PID) is an integral domain in which every ideal is principal.

Proposition 6.22. A PID is a UFD.

Corollary 6.23. An element r in a PID is prime if and only if r is irreducible.

Definition 6.24. A commutative ring with identity R is a **field** if R is an integral domain and every nonzero element $r \in R$ is a unit.

Exercise 6.25. An integral domain R is a field if and only if there is exactly one proper ideal.

Remark 6.26. If R is a field, we typically denote it by k instead of R.

Definition 6.27. A field k is **algebraically closed** if every polynomial $f \in k[x]$ has a root in k.

Definition 6.28. An ideal $\mathfrak{p} \subset R$ is a **prime** ideal if $\mathfrak{p} \subset R$ is a proper subset and whenever $r, s \in R$ with $rs \in \mathfrak{p}$, wither $r \in \mathfrak{p}$ or $s \in \mathfrak{p}$.

Exercise 6.29. An element $p \in R$ is prime if and only if (p) is a prime ideal.

Exercise 6.30. The ideal $\mathfrak{p} \subset R$ is a prime ideal if and only if $R/\mathfrak{p}$ is an integral domain.

Definition 6.31. An ideal $\mathfrak{m} \subset R$ is a **maximal** ideal if $\mathfrak{m} \subset R$ is a proper subset and if $I \subset R$ is an ideal with $\mathfrak{m} \subsetneq I \subset R$, $I = R$.

Exercise 6.32. The ideal $\mathfrak{m} \subset R$ is a maximal ideal if and only if $R/\mathfrak{m}$ is a field.

Exercise 6.33. Every maximal ideal is a prime ideal.

Proposition 6.34. If R is a PID, then a proper ideal $(f) \subset R$ is prime if and only if f is prime if and only if f is irreducible.

Proposition 6.35. If R is a PID, every nonzero prime ideal is a maximal ideal.

We conclude with the very useful correspondence theorem.

Theorem 6.36. Let $I \subset R$ be a proper ideal and let $\pi : R \to R/I$ be the natural map $\pi(r) = r + I$. There is a natural inclusion preserving bijection φ from the set of all ideals $J \subset R$ containing I and the set of all ideals in R/I given by $\varphi(J) = \pi(J)$. In addition, $\mathfrak{p} \subset R$ is prime if and only if $\varphi(\mathfrak{p})$ is prime, and $\mathfrak{m} \subset R$ is maximal if and only if $\varphi(\mathfrak{m})$ is maximal.

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