Integration Workshop 2003
Project on Harmonic Analysis
and the Distribution of Numbers

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Fejer's Theorem

Consider the circle $\mathbb{T}$ as the quotient space $\mathbb{R}/2\pi\mathbb{Z}$, and let $f\colon \mathbb{T} \to \mathbb{C}$ be a Riemann integrable function. Equivalently, f is a Riemann integrable function $f\colon \mathbb{R} \to \mathbb{C}$ with period 2π. The *Fourier coefficients* of f are defined by

$$\hat{f}(r) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-irt}\,dt = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)e^{-irt}\,dt.$$

Let $S_n(f,t)$ be given by

$$S_n(f,t) := \sum_{r=-n}^{n} \hat{f}(r)e^{irt}.$$

Fourier believed that $S_n(f,t) \to f(t)$ as $n \to \infty$ for any Riemann integrable function f. This belief is incorrect; we now know that there are many wild and crazy Riemann integrable functions that do not have this behaviour. The mathematician Du Bois-Reymond produced the first counterexample. Researchers turned to the question, "Do the Fourier coefficients determine the function f?" In other words, given $\hat{f}(r)$ for each $r \in \mathbb{Z}$, can we find $f(t)$ for $t \in \mathbb{T}$? Fejer answered this question in the affirmative and the first part of this project is to prove Fejer's result.

1. Let $\{s_n\}$ be a sequence of complex numbers. The *Cesàro limit* of s_n is given by
$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} s_k.$$
In words, the Cesàro limit is the limit, as $n \to \infty$, of the average of the first n terms of the sequence. Prove that if $s_n \to s$ as $n \to \infty$, then the Cesàro limit of s_n is also s.

2. The Cesàro limit may exist even if $\lim s_n$ does not. The sequence $\{(-1)^n\}$ does not have a limit in the usual sense, but the Cesàro limit does exist. Find the Cesàro limit of this sequence.

For a Riemann integrable function f, we define

$$\sigma_n(f,t) := \frac{1}{n+1}\sum_{j=0}^{n} S_j(f,t).$$

Hence, $\lim\limits_{n\to\infty} \sigma_n(f,t)$ is the Cesàro limit of $S_n(f,t)$.

Theorem 1 (Fejer) *Suppose $f\colon \mathbb{T} \to \mathbb{C}$ is Riemann integrable and continuous at t. Then $\lim\limits_{n\to\infty} \sigma_n(f,t) = f(t)$. Further, if f is continuous on all of $\mathbb{T}$, then the convergence is uniform.*

The next few problems will guide you through a proof of Fejer's Theorem.

3. First establish the identity

$$\sigma_n(f,t) = \sum_{r=-n}^{n} \frac{n+1-|r|}{n+1} \hat{f}(r) e^{irt}.$$

   Hint: Use induction.

4. Now, show that

$$\sigma_n(f,t) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t-y) K_n(y)\, dy,$$

   where

$$K_n(s) := \sum_{r=-n}^{n} \frac{n+1-|r|}{n+1} e^{irs}.$$

   Thus the behavior of the Cesàro sums of the Fourier series is dictated by the behavior of the integrals involving the functions $K_n(s)$.

5. Prove that if $s \neq 0$ then

$$K_n(s) = \frac{1}{n+1} \left(\frac{\sin \frac{(n+1)s}{2}}{\sin \frac{s}{2}} \right)^2 .$$

   Hint: Notice that

$$\sum_{r=-n}^{n} \Big(n+1-|r| \Big) e^{irs} = \left(\sum_{k=0}^{n} e^{i(k-n/2)s} \right)^2 .$$

6. Sketch $K_n(s)$ for a few values of n.

7. Prove that $K_n(s)$ has the following properties:

   (a) $K_n(0) = n+1$,

   (b) $K_n(s) \geq 0$ for all $s \in \mathbb{T}$,

   (c) $K_n(s) \to 0$ uniformly outside $[-\delta, \delta]$ for all $\frac{1}{4} > \delta > 0$, and

   (d) $\frac{1}{2\pi} \int_{\mathbb{T}} K_n(s)\, ds = 1$ for each $n = 0, 1, 2, \ldots.$

8. Use these properties of $K_n(s)$ to complete the proof of Fejer's Theorem.

Weyl's Equidistribution Theorem

Let γ be an irrational real number, and consider the sequence $\{2\pi r\gamma\}$ in $\mathbb{T}$. Because γ is irrational, no element of $\mathbb{T}$ appears more than once in this sequence. The following theorem says that by "sampling" a *continuous* function f on $\mathbb{T}$ at the points of this sequence, and taking a Cesàro limit, we can find the integral of f.

Theorem 2 (Weyl's Equidistribution Theorem) *Let $f\colon \mathbb{T} \to \mathbb{C}$ be a continuous function, and γ a real irrational number. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f(2\pi r\gamma) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t)\, dt.$$

Moreover, if $0 \le a \le b \le 1$, then

$$\lim_{n\to\infty} \frac{1}{n} \# \left\{ r \;\middle|\; 1 \le r \le n, \quad 2\pi r\gamma \in [2\pi a, 2\pi b] \right\} = (b-a).$$

We shall now prove this theorem. Fix an irrational real number γ. Let $C(\mathbb{T})$ denote the space of continuous functions on $\mathbb{T}$. For each $n = 1, 2, \dots$, we define an operator $G_n\colon C(\mathbb{T}) \to \mathbb{C}$ by

$$G_n(f) = \frac{1}{n} \sum_{r=1}^{n} f(2\pi r\gamma) - \frac{1}{2\pi} \int_{\mathbb{T}} f(t)\, dt.$$

9. Show that G_n is a linear transformation on $C(\mathbb{T})$.

10. For each $s \in \mathbb{Z}$, show that $\lim_{n\to\infty} G_n(e^{ist}) = 0$.

11. A *trigonometric polynomial* is any finite sum of the form

$$P(t) = \sum_{s=-m}^{m} a_s e^{ist}.$$

Show that $\lim_{n\to\infty} G_n\left(P(t)\right) = 0$ for any trigonometric polynomial $P(t)$.

12. Prove that $\lim_{n\to\infty} G_n(f) = 0$ for any continuous function f. This proves the first part of Weyl's Theorem. *Hint: Show that any continuous function on $\mathbb{T}$ can be approximated by trigonometric polynomials.*

13. Construct continuous functions f_+ and f_- such that

    (a) $f_+(t) \ge 1 \ge f_-(t)$ for all $t \in [2\pi a, 2\pi b]$.

    (b) $f_+(t) \ge 0$ for all $t \in T$, $f_-(t) = 0$ for all $t \notin [2\pi a, 2\pi b]$.

    (c) $(b-a) + \varepsilon \ge \frac{1}{2\pi} \int_{\mathbb{T}} f_+(t)\, dt$

    (d) $\frac{1}{2\pi} \int_{\mathbb{T}} f_-(t)\, dt \ge (b-a) - \varepsilon$

14. Use properties (a) and (b) to show that

$$\sum_{r=1}^{n} f_-(2\pi r\gamma) \le \# \left\{ r \;\middle|\; 1 \le r \le n, \quad 2\pi r\gamma \in [2\pi a, 2\pi b] \right\} \le \sum_{r=1}^{n} f_+(2\pi r\gamma).$$

Use this result to prove the second part of Weyl's Theorem.