Project: The space of metric spaces

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1 Hausdorff distance

One can define a metric structure on S, the closed subsets of a metric space (X, d), as follows. If F, G are closed subsets of X, then

$$d_H(F, G) = \inf\{\varepsilon : F \subset B_\varepsilon(G) \text{ and } G \subset B_\varepsilon(F)\}$$

where $B_\varepsilon(F) = \{x \in X : d(x, F) < \varepsilon\}$ and $d(x, F) = \inf\{d(x, y) : y \in F\}$.

1. Show that d_H defines a metric on S. The metric d_H is called the Hausdorff metric.

2. Show that if X is compact, then S with the metric topology generated by d_H is complete. (Hint: Let A_n be a Cauchy sequence in S. By passing to a subsequence, assume that $d_H(A_n, A_{n+1}) < 1/2^n$. Define A to be the set of all points x that are the limits of sequences $x_1, x_2, \ldots$ such that $x_i \in A_i$ and $d(x_i, x_{i+1}) < 1/2^i$. Show that $A_i \to \bar{A}$.

3. Show that if X is compact, then so is the set of closed subsets of X in the metric topology generated by d_H is totally bounded. (Hint: Use compactness to show that for any ε, X is covered by a finite set of balls of radius less than ε and show that every closed subset is in the ε ball of some subset of the set of centers of those balls.)

Conclude that if X is compact, then so is the set of closed subsets of X in the metric topology generated by d_H is complete.

2 Gromov-Hausdorff distance

Recall that if (X, d_X) and (Y, d_Y) are metric spaces, an isometric embedding is an injective map $\Phi : X \to Y$ such that $d_Y(\Phi(x), \Phi(x')) = d_X(x, x')$.

We can define a metric on the set of compact metric spaces as follows. Given (X, d_X) and (Y, d_Y) define the Gromov-Hausdorff distance between them as

$$d_{GH}(X, Y) = \inf\{d_H(\Phi(X), \Psi(Y))\}$$

where d_H is the Hausdorff distance on the closed subsets of some metric space Z and the inf is taken over all isometric embeddings $\Phi : X \to Z$ and $\Psi : Y \to Z$.

1. Show that d_{GH} is a metric on the space of compact metric spaces modulo isometries.

2. We can define a different metric d'_{GH} as follows. Define an ε-approximate isometry to be a (not-necessarily continuous) map $\phi : X \to Y$ such that $B_\varepsilon(\phi(X)) = Y$ and $|d_Y(\phi(x), \phi(x')) - d_X(x, x')| < \varepsilon$ for all $x, x' \in X$. Define d'_{GH} as

   $$d'_{GH}(X, Y) = \inf\{\varepsilon : \text{there exist } \varepsilon\text{-approx. isometries } X \to Y \text{ and } Y \to X\}.$$

   Show that d_{GH} and d'_{GH} generate the same topology.

3. Show that the sequence of circles of radius $1/n$ converge to a point in the Gromov-Hausdorff topology. This simple example illustrates why the Gromov-Hausdorff topology is so interesting. It allows two spaces to be close even if they are not homeomorphic.

4. Show that for any metric space X and any $\varepsilon > 0$ there is a metric on a finite set of points F such that $d_{GH}(X, F) < \varepsilon$. Hence metrics on finite point sets are dense with respect to the Gromov-Hausdorff topology. F is called an ε-net.

5. For compact metric spaces X and $\{X_n\}_{n=1}^\infty$, we have that $X_n \to X$ in the Gromov-Hausdorff topology if and only if for every $\varepsilon > 0$ there exists a finite ε-net S in X and ε-nets S_n in each S such that $S_n \to S$ in the Gromov-Hausdorff topology. Hence convergence of metric spaces in Gromov-Hausdorff topology is equivalent to convergence of ε-nets.

6. Theorem: Let C be a collection of compact metric spaces such that (1) there is a constant D such that the diameter of X is bounded by D for all $X \in C$ and (2) for every $\varepsilon > 0$ there exists a natural number $N = N(\varepsilon)$ such that for every $X \in C$ there is a set of points $x_1, \ldots, x_N \in X$ such that $X = \bigcup_{i=1}^{N} B(x_i, \varepsilon)$ (note that these balls are not required to be disjoint). Then C is precompact in the Gromov-Hausdorff topology, i.e. any sequence in C has a subsequence which converges (though the limit may not remain in C).

   We step through this proof as follows: Consider a sequence $\{X_n\}_{n=1}^\infty$ of spaces in C.

   (a) Show that for each k and n, there exists a finite set of points x_i which form a $1/k$-net in X_n. Now show there is a sequence $\{x_{i,n}\}_{i=1}^\infty$ consisting of the the union of the x_i for $k = 1, 2, 3, \ldots$ (here we have a different sequence for each n). Show that $\{x_{i,n}\}_{i=1}^\infty$ is a countable, dense subset of X_n (recall a set is dense in X if its closure is all of X) and there exists $M(k) = M(k-1) + N(1/k)$ (take $M(1) = 1$) independent of n such that $\{x_{i,n}\}_{i=1}^{M(k)}$ form a $1/k$-net. (Hint: use condition (2) in the theorem.)

(b) Show that there is a subsequence of $\{X_n\}$ such that $\{d(x_{i,n}, d_{j,n})\}_{n=1}^{\infty}$ converges for all i, j.

(c) Reindex the subsequence so that it is once again indexed by n. Define a metric space on the abstract countable set $\{x_i\}_{i=1}^{\infty}$ in the following way. Define $d(x_i, x_j)$ by

$$d(x_i, x_j) = \lim_{n\to\infty} d(x_{i,n}, x_{j,n}).$$

This may not be a metric because $d(x_i, x_j)$ may equal zero when $i \neq j$. Define the metric space by quotienting out by the relation that $x_i \sim x_j$ if $d(x_i, x_j) = 0$. Show this is a metric space. We will denote this quotient space by $\tilde{X}$ and elements in the quotient by $\tilde{x}_i$

(d) For each k, consider the sets $S^k = \{\tilde{x}_i : 1 \leq i \leq M(k)\}$ and $S_n^k = \{\tilde{x}_{i,n} : 1 \leq i \leq M(k)\}$. We know that S_n^k is a $1/k$-net. Show that for each i there is a $j \leq M(k)$ such that $d(x_{i,n}, x_{j,n}) < 1/k$ for infinitely many n. Conclude that S^n is a $1/k$-net in $\tilde{X}$.

(e) Show that X_n converges to $\tilde{X}$. Show that $\tilde{X}$ is totally bounded, so its completion is compact and we are done. (We do not go into the process of completion, but rest assured that every countable metric space has a completion. It is defined as an equivalence class of Cauchy sequences.)

It is a corollary that the set of all Riemannian manifolds with Ricci curvature uniformly bounded below and diameter uniformly bounded above are precompact (the closure is compact) in the Gromov-Hausdorff topology.