

MULTILINEAR ALGEBRA

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1. INTRODUCTION

This project consists of a rambling introduction to some basic notions in multilinear algebra. The central purpose will be to show that the div, grad, and curl operators from vector calculus can all be thought of as special cases of a single operator on differential forms. Not all of the material presented here is required to achieve that goal. However, all of these ideas are likely to be useful for the geometry core course.

Note: I will use angle brackets to denote the pairing between a vector space and its dual. That is, if V is a vector space and V^* is its dual, for any $v \in V$, $\phi \in V^*$, instead of writing $\phi(v)$ for the result of applying the function ϕ to v , I will write $\langle \phi, v \rangle$. The motivation for this notation is that, given an inner product $\langle \cdot, \cdot \rangle$ on V , every element $\phi \in V^*$ can be written in the form $v \mapsto \langle v_0, \cdot \rangle$ for some vector $v_0 \in V$. The dual space V^* can therefore be identified with V (via $\phi \mapsto v_0$). The reason we use the term, *pairing*, is that duality of vector spaces is symmetric. By definition, ϕ is a linear function on V , but v is also a linear function on V^* . So instead of thinking of applying ϕ to v , we can just as well consider v to be acting on ϕ . The result is the same in both cases. For this reason, it makes more sense to think of a *pairing* between V and V^* : If you pair an element from a vector space with an element from its dual, you get a real number. This notation will also be used for multilinear maps.

2. PRODUCTS OF VECTOR SPACES

2.1. Tensor Products.

Definition 2.1. Suppose V and W are vector spaces with bases $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$, respectively. Then $V \otimes W$ is the formal span over \mathbb{R} of the elements $\{v_i \otimes w_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$.

In other words, an element of $V \otimes W$ has the form

$$\sum_{1 \leq i \leq n, 1 \leq j \leq m} c_{ij}(v_i \otimes w_j),$$

for some constants c_{ij} . This definition depends on the bases chosen for V and W , but it is obvious that a different pair of choices yields an isomorphic result.

Alternatively,

Definition 2.2. Let $V \otimes W$ be the set of elements $\{v \otimes w \mid v \in V, w \in W\}$ modulo the following equivalence relations:

$$\begin{aligned} v \otimes (w_1 + w_2) &= v \otimes w_1 + v \otimes w_2 \\ (v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w \\ (cv) \otimes w = v \otimes (cw) &= c(v \otimes w). \end{aligned}$$

Exercise 2.3. Show that definitions 2.1 and 2.2 are equivalent.

Now suppose V is a vector space and V^* is its dual. The elements of $V^* \otimes V^*$ can be identified with real-valued functions on $V \times V$ as follows: If $\phi \otimes \psi \in V^* \otimes V^*$ and $v, w \in V$, then

$$\langle \phi \otimes \psi, (v, w) \rangle := \langle \phi, v \rangle \cdot \langle \psi, w \rangle.$$

This action is then extended linearly to arbitrary elements of $V^* \otimes V^*$.

Exercise 2.4. Show that with the action described above, the elements of $V^* \otimes V^*$ are *bilinear* maps on $V \times V$. That is, show that for $\phi \otimes \psi \in V^* \otimes V^*$, $(v_1, w_1), (v_2, w_2) \in V \times V$ and $a_1, b_1, a_2, b_2 \in \mathbb{R}$,

$$\begin{aligned} \langle \phi \otimes \psi, (a_1 v_1 + a_2 v_2, b_1 w_1 + b_2 w_2) \rangle &= a_1 b_1 \langle \phi \otimes \psi, (v_1, w_1) \rangle \\ &+ a_1 b_2 \langle \phi \otimes \psi, (v_1, w_2) \rangle \\ &+ a_2 b_1 \langle \phi \otimes \psi, (v_2, w_1) \rangle \\ &+ a_2 b_2 \langle \phi \otimes \psi, (v_2, w_2) \rangle \end{aligned}$$

Exercise 2.5. Show that $V^* \otimes V^*$ is the space of *all* multilinear maps on $V \times V$.

2.2. Wedge Products. Of particular interest in differential geometry is the subspace $V^* \wedge V^* \subset V^* \otimes V^*$ of so-called *alternating tensors*. For any $\phi, \psi \in V^*$, define the element $\phi \wedge \psi \in V^* \otimes V^*$ by its action on elements of the form $v \otimes w \in V \otimes V$:

$$\langle \phi \wedge \psi, v \otimes w \rangle := \det \begin{pmatrix} \langle \phi, v \rangle & \langle \psi, v \rangle \\ \langle \phi, w \rangle & \langle \psi, w \rangle \end{pmatrix} = \langle \phi, v \rangle \cdot \langle \psi, w \rangle - \langle \phi, w \rangle \cdot \langle \psi, v \rangle.$$

Of course, typical elements of $V \otimes V$ have the form $\sum_i v_i \otimes w_i$, and similarly for $V^* \otimes V^*$, so the above definition must be extended by linearity.

Notice that interchanging v and w interchanges the rows and interchanging ϕ and ψ interchanges the columns of the above matrix. Since we are taking the determinant, this has the effect of multiplying the result by -1 . Linear maps on $V \otimes V$ with this property are called *alternating 2-tensors*.

Exercise 2.6. Prove that the space of all alternating 2-tensors on V is the vector space $V^* \wedge V^* = \{\phi \wedge \psi \mid \phi, \psi \in V^*\}$.

Exercise 2.7. Prove that if $\{\phi_1, \dots, \phi_n\}$ is a basis for V^* , then $\{\phi_i \wedge \phi_j \mid 1 \leq i < j \leq n\}$ is a basis for $V^* \wedge V^*$.

Exercise 2.8. Suppose $V = \text{span} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}$, and $V^* = \text{span} \{dx, dy, dz\}$. Write expressions in terms of these bases for generic elements of $V \wedge V$ and $V^* \wedge V^*$.

Note that if V and W are different vector spaces, then $V \otimes W$ makes sense, but $V \wedge W$ does not.

In general, for $\phi_1, \dots, \phi_n \in V^*$ and $v_1, \dots, v_n \in V$, we define $\phi_1 \wedge \dots \wedge \phi_n$ by its action on $\otimes^n V$, the n -fold tensor product of V with itself:

$$\langle \phi_1 \wedge \dots \wedge \phi_n, (v_1, \dots, v_n) \rangle := \det (\langle \phi_j, v_i \rangle) = \det \begin{pmatrix} \langle \phi_1, v_1 \rangle & \dots & \langle \phi_n, v_1 \rangle \\ \vdots & & \vdots \\ \langle \phi_1, v_n \rangle & \dots & \langle \phi_n, v_n \rangle \end{pmatrix}.$$

Again, this action is extended by linearity. As before, interchanging v_i with v_j or ϕ_i and ϕ_j changes the sign of the result. More generally, any permutation of the ϕ s or the v s can

be decomposed into a product of transpositions (interchanges of adjacent elements). For a given permutation, the parity of the number of transpositions in this decomposition is called the *sign* of the permutation. Thus, an even permutation of the ϕ s or v s leaves the result unchanged, and an odd permutation changes the sign of the result. This is what it means to be alternating in the case of n -fold tensors. For example, suppose $n = 3$, and let $\phi_1, \phi_2, \phi_3 \in V^*$, $v_1, v_2, v_3 \in V$. Then

$$\langle \phi_1 \wedge \phi_2 \wedge \phi_3, v_1 \otimes v_3 \otimes v_2 \rangle = -\langle \phi_1 \wedge \phi_2 \wedge \phi_3, v_1 \otimes v_2 \otimes v_3 \rangle$$

because $v_1 \otimes v_3 \otimes v_2$ and $v_1 \otimes v_2 \otimes v_3$ differ by a single transposition. On the other hand,

$$\langle \phi_1 \wedge \phi_2 \wedge \phi_3, v_3 \otimes v_1 \otimes v_2 \rangle = \langle \phi_1 \wedge \phi_2 \wedge \phi_3, v_1 \otimes v_2 \otimes v_3 \rangle$$

since $v_3 \otimes v_1 \otimes v_2$ and $v_1 \otimes v_2 \otimes v_3$ differ by a pair of transpositions. As noted, the results would be the same if the ϕ s were permuted instead of the v s. Note, also, that if V is n -dimensional and $k > n$, then $\wedge^k V$ is necessarily 0. Finally, $\wedge^0(V)$ is simply the set of real numbers.

Exercise 2.9. Let $\{\phi_1, \dots, \phi_n\}$ be a basis for V^* . Show that $\{\phi_{i_1} \wedge \phi_{i_2} \wedge \dots \wedge \phi_{i_k} \mid 1 \leq i_1 < \dots < i_k \leq n\}$ is a basis for $\wedge^k(V^*)$.

Exercise 2.10.

1. Show that $\wedge^k(V^*)$ is the space of *all* alternating, k -linear maps on V .
2. Give two other descriptions of $\wedge^k(V^*)$: one as a set of alternating, linear functions and another as a set of linear functions.

Exercise 2.11. Show that the wedge product is associative. That is, show that for $\alpha, \beta, \gamma \in V$, $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$.

Exercise 2.12.

1. Show that for any element of $\alpha \in \otimes^2(\mathbb{R}^n)$, there exists a matrix X_α such that $\langle \alpha, (\vec{v}, \vec{w}) \rangle = \vec{v}^T X_\alpha \vec{w}$ for all $\vec{v}, \vec{w} \in \mathbb{R}^n$.
2. Show that α is alternating if and only if X_α is skew-symmetric.
3. Compute the matrix for the standard inner product on \mathbb{R}^n .
4. Define a pairing \langle, \rangle on the vector space

$$\mathfrak{sl}(2, \mathbb{C}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \mid \alpha, \beta, \gamma \in \mathbb{C} \right\}$$

by

$$\langle A, B \rangle = \Im \text{tr}(AB)$$

(the imaginary part of the trace of the product). Find the matrix corresponding to this bilinear form. Find the signature of this form.

3. TENSOR FIELDS ON \mathbb{R}^n

To each point $p \in \mathbb{R}^n$, we will associate a copy of \mathbb{R}^n called the *tangent space* at p , denoted by $T_p \mathbb{R}^n$. Given coordinates x_1, \dots, x_n on \mathbb{R}^n , we will use the basis $\left\{ \left(\frac{\partial}{\partial x_1} \right)_p, \dots, \left(\frac{\partial}{\partial x_n} \right)_p \right\}$ for $T_p \mathbb{R}^n$, where $\left(\frac{\partial}{\partial x_i} \right)_p$ can be thought of as a unit vector, based at p and pointing in the

(positive) x_i direction. The dual of $T_p\mathbb{R}^n$ is called the *cotangent space* at p , and is denoted by $T_p^*\mathbb{R}^n$. The dual basis to $\left\{\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p\right\}$ is denoted by $\{dx_1, \dots, dx_n\}$.

3.1. Multivector Fields. A vector field on \mathbb{R}^n is a function X which picks out from each tangent space T_pM a vector X_p in such a way that the vectors X_p “vary smoothly” as you move around the \mathbb{R}^3 . A vector in T_pM looks like

$$\sum_{i=1}^n c_i \left(\frac{\partial}{\partial x_i}\right)_p,$$

for some constants c_1, \dots, c_n , and a vector field on \mathbb{R}^3 has the form

$$\sum_{i=1}^n f_i(p) \left(\frac{\partial}{\partial x_i}\right)_p,$$

where f_1, f_2, f_3 are functions from \mathbb{R}^3 to \mathbb{R} . “Varying smoothly” just means that the coefficient functions f_1, f_2 , and f_3 are smooth. In practice, the p subscripts are often omitted, and the f_i ’s are written in x, y, z coordinates. For example,

$$V = \sin(y) \frac{\partial}{\partial x} - x^2 \frac{\partial}{\partial y} + 2 \frac{\partial}{\partial z}$$

is a vector field on \mathbb{R}^3 .

More generally, a k -vector field (or multivector field) on \mathbb{R}^n assigns to each point $p \in \mathbb{R}^n$ an element of the vector space $\wedge^k(T_p\mathbb{R}^n)$. For example,

$$\pi = z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$$

is a 2-vector field (often called a bivector field) on \mathbb{R}^3 .

3.2. Differential Forms. The dual object to a multivector field is a *differential form*: A differential k -form (or just a k -form) on \mathbb{R}^n is a function ω which assigns to each point $p \in \mathbb{R}^n$ an element ω_p of $\wedge^k(T_p^*\mathbb{R}^n)$. The wedge product of two forms is a pointwise operation: if ω is a k -form and η is an l -form, then $\omega \wedge \eta$ is a $(k+l)$ -form defined by

$$(\omega \wedge \eta)_p := \omega_p \wedge \eta_p.$$

A 0-form assigns a real number to each point in \mathbb{R}^n . That is, a 0-form is a smooth function on \mathbb{R}^n . If f is a 0-form and ω is a k -form, then $f \wedge \omega = f \cdot \omega$.

Exercise 3.1. Using coordinates $\{x_1, \dots, x_5\}$, find expressions for general 1-forms, 2-forms, 3-forms, 4-forms, and 5-forms on \mathbb{R}^5 .

Exercise 3.2.

1. Let I denote a multi-index (i_1, \dots, i_n) where $1 \leq i_1 \leq \dots \leq i_n \leq n$, and let dx_I denote the k -form $dx_{i_1} \wedge \dots \wedge dx_{i_n}$. Show that any k -form ω on \mathbb{R}^n can be represented as $\omega = \sum_I f_I dx_I$ for some functions f_I .
2. Compute the dimension of $\Omega^k(\mathbb{R}^n)$, the vector space of k -forms on \mathbb{R}^n .

Exercise 3.3. Show that if ω is a k -form and η is an l -form, then $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$.

4. THE d -OPERATOR AND VECTOR CALCULUS

In this section, we will show that the div , grad , and curl operators from vector calculus can be interpreted as special cases of a single operator on differential forms.

4.1. **The d -operator.** The d -operator on \mathbb{R}^n will map k -forms to $(k + 1)$ -forms.

Definition 4.1. For a 0-form f (i.e., a function), set $df := \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$. For a k -form $\omega = \sum_I f_I dx_I$, set

$$d\omega := \sum_I df_I \wedge dx_I.$$

Exercise 4.2.

1. Show that the d -operator is linear.
2. Show that if ω is a k -form and η is an l -form,

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge (d\eta).$$

Exercise 4.3. Let $\omega = zdx \wedge dy + ydz \wedge dx + xdy \wedge dz$ be a 3-form on \mathbb{R}^3 . Compute $d\omega$.

Exercise 4.4. Show that for any k -form ω , $d(d\omega) = 0$. More succinctly, $d^2 = 0$.

4.2. **Vector Calculus: grad , curl , and div .** On \mathbb{R}^3 , we have

- $\text{grad} : \text{functions} \rightarrow \text{vector fields}$,
- $\text{curl} : \text{vector fields} \rightarrow \text{vector fields}$,
- $\text{div} : \text{vector fields} \rightarrow \text{functions}$.

Exercise 4.5. Any element $\omega_{n-k} \in \Omega^{n-k}(\mathbb{R}^n)$ can be thought of as a map from $\Omega^k(\mathbb{R}^n)$ to $\Omega^n(\mathbb{R}^n)$ by

$$\omega_{n-k}(\omega_k) := \omega_{n-k} \wedge \omega_k.$$

1. Explain why it follows that $(\Omega^k(\mathbb{R}^n))^* \cong \Omega^{n-k}(\mathbb{R}^n)$.
2. Find the basis for $\Omega^2(\mathbb{R}^3)$ that is dual to the basis $\{dx, dy, dz\}$ for $\Omega^1(\mathbb{R}^3)$ according to the identification in part (1).

Exercise 4.6. Define maps

- $T_1 : 1\text{-forms} \rightarrow \text{functions}$,
- $T_2 : 2\text{-forms} \rightarrow \text{vector fields}$, and
- $T_3 : 3\text{-forms} \rightarrow \text{functions}$

such that if ω_0 is a 0-form, ω_1 is a 1-form, and ω_2 is a 2-form,

$$\begin{aligned} T_1(d\omega_0) &= \text{grad}(T_0(\omega_0)), \\ T_2(d\omega_1) &= \text{curl}(T_1(\omega_1)), \text{ and} \\ T_3(d\omega_2) &= \text{div}(T_2(\omega_2)). \end{aligned}$$

Comments:

1. For these maps to fulfill the goals of this section, they should be bijective.

2. The maps T_1 and T_3 are very natural. One way to motivate T_2 is via the identification in exercise 4.5.

Exercise 4.7.

1. Show that $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$ for any vector field \vec{F} on \mathbb{R}^3 .
2. Show that $\operatorname{curl}(\operatorname{grad}f) = 0$ for any smooth function f on \mathbb{R}^3 .