Integration Workshop 2003
Project on Modules over Principal Ideal Domains

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In this project we develop a proof of the fundamental structure theorem for modules over principal ideal domains which is different from the one given in the lectures.

1. Let M be a module over an integral domain R. An element $x \in M$ is called a torsion element if $rx = 0$ for some nonzero element $r \in R$. Let T be the set of all torsion elements of M.
   (a) Show that T is a submodule of M. (It is called the torsion submodule of M.)
   (b) Show that the quotient module M/T is torsion-free; that is, the only torsion element in M/T is 0.
2. Let M be an R-module. Prove that the *annihilator*
$$A = \{r \in R | rx = 0, \forall x \in M\}$$
   of M is an ideal in R.
3. From now on suppose that R is a principal ideal domain. Show that a finitely generated torsion-free module over R is free (Hint: this is not as easy as it sounds).
4. Show that if $A = (r)$ is the annihilator for a module M, then there exists $x_0 \in M$ such that $A = \{r \in R | rx_0 = 0\}$. (Hint: Use the fact that every PID is a UFD.)
5. Show that, with x as above, there exists a direct sum decomposition $M = N \oplus Rx_0$. (Here Rx_0 is the cyclic submodule of M generated by x_0.)
6. Prove that if M is a finitely generated torsion module over a PID R, then $M = M_1 \oplus M_2 \oplus \ldots \oplus M_k$ where each M_i is cyclic (generated by one element; i.e., $M_i = Rx_i$) with annihilator $A_i = (r_i)$ with $r_i | r_{i-1}, 2 \leq i \leq k$.
7. Suppose R is a ring with unity, M is an R-module, F is a free R-module, and we have a function f of R-modules $f : M \to F$ that is onto. Prove that there exists a submodule E of M such that E is isomorphic to F, and $M = E \oplus ker(f)$.

8. Conclude that, for any module M over a PID R, there exist non-negative integers m, k and non-zero, non-units $r_1, \ldots, r_k \in R$ with $r_i | r_{i-1}$ for $2 \leq i \leq k$ such that

$$M \cong R/(r_1) \oplus R/(r_2) \oplus \ldots \oplus R/(r_k) \oplus R^m$$

9. To what extent are the integers m and the elements r_i unique?