INTEGRATION WORKSHOP 2006
MÖBIUS TRANSFORMATIONS OF THE COMPLEX UPPER HALF-PLANE

NICK ROGERS

Define $SL_2(\mathbb{R})$ to be the group of 2×2 matrices over $\mathbb{R}$ with determinant 1. Let $SL_2(\mathbb{R})$ act on the upper half-plane

$$\mathcal{H} = \{z \in \mathbb{C} : \mathfrak{Im}(z) > 0\}$$

via Möbius transformations; that is, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\gamma z = \dfrac{az+b}{cz+d}$. The main goal of this project will be to prove the following:

Theorem. A subgroup $G \subset PSL_2(\mathbb{R})$ acts properly discontinuously on $\mathcal{H}$ if and only if G is discrete.

Verify that Möbius transformations define a transitive group action, and show that the kernel of the action is the subgroup $\{\pm I\}$. To check that $\gamma z \in \mathcal{H}$ if $z \in \mathcal{H}$, prove and then apply the following identity:

$$\mathfrak{Im}\left(\frac{az+b}{cz+d}\right) = \frac{(ad-bc)\mathfrak{Im}(z)}{|cz+d|^2}. \tag{1}$$

The quotient of $SL_2(\mathbb{R})$ by $\{\pm I\}$ is called $PSL_2(\mathbb{R})$. We often represent elements of $PSL_2(\mathbb{R})$ as matrices, even though elements of $PSL_2(\mathbb{R})$ are really 2-element equivalence classes of matrices. Endow $SL_2(\mathbb{R})$ with the subspace topology inherited from $\mathbb{R}^4$, and give $PSL_2(\mathbb{R})$ the quotient topology. Show that that the map $PSL_2(\mathbb{R}) \times \mathcal{H} \to \mathcal{H}$ associated to the group action is continuous.

A group G that is also a topological space is called a *topological group* if the maps

$$\begin{array}{lcl} \mu : G \times G \to G & \text{and} & \iota : G \to G \\ (\gamma_1, \gamma_2) \mapsto \gamma_1\gamma_2 & & \gamma \mapsto \gamma^{-1} \end{array}$$

are continuous. Show that $PSL_2(\mathbb{R})$ is a topological group.

One may similarly define $SL_2(\mathbb{Z})$ to be the set of 2×2 matrices with entries in $\mathbb{Z}$ and determinant 1. Check that $SL_2(\mathbb{Z})$ is a subgroup of $SL_2(\mathbb{R})$, and define $PSL_2(\mathbb{Z})$ to be the quotient of $SL_2(\mathbb{Z})$ by $\{\pm I\}$. Show that a topological group has the discrete topology if and only if the singleton set consisting of the identity element is open. Use this to show that $PSL_2(\mathbb{Z})$ is a discrete subgroup of $PSL_2(\mathbb{R})$. Subgroups of topological groups are always topological groups under the subspace topology, but of course any group with the discrete topology is automatically a topological group.

An action of a group G on a topological space X is said to be *properly discontinuous* (or, sometimes, simply *discontinuous*) if every point $x \in X$ has a neighborhood U_x so that the set

$$\{\gamma \in G : \gamma U_x \cap U_x \neq \varnothing\}$$

is finite. Be careful! Discontinuous is not the opposite of continuous in this context.

Show that any discrete subgroup of $PSL_2(\mathbb{R})$ acts properly discontinuously on $\mathcal{H}$ as follows:

- For $z \in \mathcal{H}$, let U_z be a bounded neighborhood of z such that, for some $\epsilon > 0$, $\mathfrak{Im}(w) > \epsilon$ for all $w \in U_z$. Use equation (1) to show that the set
$$\{(c,d) \in \mathbb{R}^2 : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) U_z \cap U_z \neq \varnothing \text{ for some } a,\, b\}$$
is bounded.
- Suppose that γ_1 and γ_2 have the same second row, and compute $\gamma_1\gamma_2^{-1}$. How does a matrix of this form act on $\mathcal{H}$?
- Conclude that the set
$$\{\gamma \in PSL_2(\mathbb{R}) : \gamma U_z \cap U_z \neq \varnothing\}$$
is bounded, and use the Heine-Borel theorem to conclude that the intersection of this set with any discrete subgroup of $PSL_2(\mathbb{R})$ is finite.

A family $\{M_\alpha\}$ of subsets of a topological space X is called *locally finite* if, for any compact subset $K \subset X$, $M_\alpha \cap K \neq \varnothing$ for only finitely many α. Prove that an action of G on X is properly discontinuous if and only if the family $\{\gamma x : \gamma \in G\}$ is locally finite for each $x \in X$. Note that if a point $x \in X$ has nontrivial stabilizer, then each point in the orbit Gx of x appears in this family with multiplicity > 1.

Conversely, suppose that a subgroup G of $PSL_2(\mathbb{R})$ acts properly discontinuously on the upper half-plane. Use the continuity of the action to show that there is an open neighborhood of the identity consisting of elements that simultaneously stabilize every $z \in \mathcal{H}$, and conclude that G is a discrete group.

Show that if G is a discrete subgroup of $PSL_2(\mathbb{R})$, then there is an open subset F of $\mathcal{H}$ such that:

(i) For all $z \in \mathcal{H}$, there exists a $\gamma \in G$ such that $\gamma z \in \overline{F}$, the closure of F.
(ii) If $z_1, z_2 \in F$ and $\gamma z_1 = z_2$ for some $\gamma \in G$, then $z_1 = z_2$ and $\gamma = I$.

Such an F is called a *fundamental domain* for G.

Show that if $G = PSL_2(\mathbb{Z})$, then one can take as the fundamental domain
$$F = \{z \in \mathcal{H} : -\tfrac{1}{2} < \mathfrak{Re}(z) < \tfrac{1}{2},\ |z| > 1\} :$$

- For property (i), use equation (1) to show that there is a matrix $\gamma_0 \in PSL_2(\mathbb{Z})$ that maximizes $\mathfrak{Im}(\gamma_0 z)$. Next show that there is some n for which $T_n = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ satisfies $-\frac{1}{2} \leq \mathfrak{Re}(T_n\gamma_0 z) \leq \frac{1}{2}$. Finally, apply the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ to conclude that $T_n\gamma_0 z \in \overline{F}$.
- For property (ii), argue that we may assume $\mathfrak{Im}(\gamma z) \geq \mathfrak{Im}(z)$, and then use equation (1) to give a finite list of values of c that may occur in the matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Analyze each of these values of c separately.

By construction, $\gamma F \cap F = \varnothing$ for all $\gamma \in PSL_2(\mathbb{Z})$. However, this is not true if we replace F with $\overline{F}$. Which elements of $PSL_2(\mathbb{Z})$ take parts of the boundary of F to itself? Are there any points on the boundary of F with nontrivial stabilizer in $PSL_2(\mathbb{Z})$? Sketch the region F, and show how parts of the boundary are identified under the action of $PSL_2(\mathbb{Z})$.

The topological space obtained by identifying points of $\mathcal{H}$ under the action of $PSL_2(\mathbb{Z})$ is called $Y(1)$, and its one-point compactification is called $X(1)$. They are intimately related to the structure of the set of *elliptic curves*, which are curves defined by equations such as $y^2 = x^3 + ax + b$, and form a very important area of research in modern number theory.