INTEGRATION WORKSHOP 2004
PROJECT ON TOPOLOGICAL VECTOR SPACES

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A (real) *topological vector space* is a vector space V over $\mathbb{R}$ with a Hausdorff topology such that the addition and scalar multiplication maps $V \times V \to V$ and $\mathbb{R} \times V \to V$ are continuous.

Prove that a linear map $V \to W$ of topological vector spaces is continuous if and only if it is continuous at the origin.

If V is a finite dimensional vector space and we fix a vector space isomorphism $\phi : \mathbb{R}^n \to V$, then we may transfer the metric topology from $\mathbb{R}^n$ to V: the open subsets of V are the sets $\phi(U)$ where U is open in $\mathbb{R}^n$. Prove that the resulting topology on V is independent of the choice of ϕ.

More generally, there is only one topology on a finite dimensional vector space V which makes it into a topological vector space. Hints for a proof: Choose a vector space isomorphism $\phi : \mathbb{R}^n \to V$. Define $S = \{x \in \mathbb{R}^n | |x| = 1\}$ and $B = \{x \in \mathbb{R}^n | |x| < 1\}$.

(1) Prove directly from the definitions that ϕ is continuous.
(2) Prove that $\phi(S)$ is closed in V and so $V \setminus \phi(S)$ is open.
(3) Use the continuity of $\mathbb{R} \times V \to V$ to show that every neighborhood of the origin in V contains an open set U such that $0 \in U$ and $\alpha U \subset U$ for all real $-1 < \alpha < 1$.
(4) Apply the previous step to $V \setminus \phi(S)$ to prove that ϕ^{-1} is continuous.

What happens if we drop the Hausdorff assumption?

Given an example of an infinite dimensional vector space V with two different topologies $\mathcal{U}$ and $\mathcal{U}'$ (i.e., such that the identity map $V \to V$ does not induce a homeomorphism $(V, \mathcal{U}) \to (V, \mathcal{U}')$). Can you prove that *no* linear map $V \to V$ induces a homeomorphism $(V, \mathcal{U}) \to (V, \mathcal{U}')$?