Integration Workshop 2005
Project on Banach Algebras

Feryâl Alayont

A Banach algebra is a complete normed vector space which is also a ring. All finite dimensional algebras over **R** or **C**, for instance, the matrix algebras, are Banach algebras. Typical infinite dimensional examples are the algebra of operators on a Banach space and the algebra of continuous functions on a compact Hausdorff space.

An algebraic result proved by Frobenius in 1878 says that the only finite dimensional real division algebras, i.e. the real algebras in which non-zero elements are invertible, are **R**, **C** and the quaternions **H**. As a corollary, the only possible finite dimensional complex division algebra is **C** itself.

In 1938, Mazur announced that these are also the only real Banach algebras which are division algebras. The proof used the theory of harmonic functions. In 1941, Gelfand gave an elegant proof of the theorem for the case of complex Banach algebras using complex analysis. In this project we will prove this result, now known as the Gelfand-Mazur Theorem.

Let K be **R** or **C**. A K-*algebra* (with identity) is a K-vector space A in which a multiplication is defined that satisfies

i. $x(yz) = (xy)z$ for all $x, y, z \in A$ (associativity),

ii. $(x + y)z = xz + yz$ and $x(y + z) = xy + xz$ for all $x, y, z \in A$ (distributivity),

iii. $\alpha(xy) = (\alpha x)y = x(\alpha y)$ for all $x, y, z \in A$ and $\alpha \in K$,

iv. there exists an $e \in A$ such that $xe = ex = x$ for all $x \in A$.

A is called *commutative* if $xy = yx$ for all $x, y \in A$.

Examples: • K^n with coordinate-wise multiplication is a commutative K-algebra.

• $\mathbf{C}^n$, when viewed as a real vector space, is a commutative real algebra.

• Let S be a non-empty compact Hausdorff space and $C(S, K)$ be the space of all continuous K-valued functions on S. Define multiplication in $C(S, K)$ as follows:

$$(f \cdot g)(x) = f(x)g(x)\,.$$

Then $C(S, K)$ is a commutative K-algebra.

• Let $M_n(K)$ be the set of $n \times n$ matrices with entries in K, $n > 1$. Then $M_n(K)$ is a non-commutative K-algebra.

Recall that a *norm* on a K-vector space V is a function $\|\cdot\| : V \to \mathbf{R}$ such that

i. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$ (Triangle Inequality),

ii. $\|\alpha x\| = |\alpha|\|x\|$ if $x \in V$ and $\alpha \in K$,

iii. $\|x\| > 0$ if $x \neq 0$.

A *Banach space* is a vector space with a norm which is *complete* in the metric defined by this norm, i.e. every Cauchy sequence converges to a point in the space.

Examples: • K^n with the supremum norm

$$\|\mathbf{x}\| = \|(x_1, \ldots, x_n)\| = \sup\{|x_1|, \ldots, |x_n|\}$$

is a Banach space.

• $C(S, K)$ with the supremum norm

$$\|f\| = \sup_{s \in S} |f(s)| \ .$$

is a Banach space.

Exercise: Let $\| \cdot \|_2$ denote the Euclidean norm on $\mathbf{R}^n$. Define a norm on $M_n(\mathbf{R})$ by

$$\|A\| = \sup_{\|v\|_2 = 1} \|Av\|_2 \ .$$

Show that $\|A\|$ is the maximum of $|\lambda|^{1/2}$ as λ ranges over the eigenvalues of $A^T A$ and that $M_n(\mathbf{R})$ is a Banach space with this norm.

Hint: Note that $\|Ax\| = x^T A^T Ax$ and that there exists an orthonormal basis of $\mathbf{R}^n$ consisting of eigenvectors of $A^T A$. For the second claim, note that $|a_{ij}| \leq \|A\|$ for all i, j.

Note: A similar result holds for $M_n(\mathbf{C})$. Change the transposes to conjugate transposes.

If a K-algebra A is a Banach space with respect to a norm that satisfies

$$\|xy\| \leq \|x\| \, \|y\| \qquad \text{and} \qquad \|e\| = 1 \ ,$$

then A is called a K-Banach algebra.

Exercises: Show that

• K^n with the supremum norm and coordinate-wise multiplication is a K-Banach algebra.

• $C(S, K)$ with the supremum norm and pointwise multiplication is a K-Banach algebra.

• $M_n(K)$ with the operator norm is a K-Banach algebra.

Now that we have some familiarity with Banach algebras, we can start proving some results in general Banach algebras.

Let A be a Banach algebra. An element $x \in A$ is said to be *invertible* if there exists a $y \in A$ such that $xy = yx = e$. Let $G(A)$ be the set of invertible elements. Then $G(A)$ is a group. We want to show that $G(A)$ is open in A.

Lemma 1 *Let $x \in A$, $\|x\| \leq 1$. Then $e - x$ is invertible and*

$$\|(e - x)^{-1} - e\| \leq \frac{\|x\|}{1 - \|x\|} \ .$$

Hint: $x_n = e + x + x^2 + \ldots + x^n$ is a Cauchy sequence in A.

Lemma 2 *Let $x \in G(A)$, $h \in A$ and $\|h\| \leq \frac{1}{2}\|x^{-1}\|^{-1}$. Then $x + h \in G(A)$ and*

$$\|(x + h)^{-1} - x^{-1}\| \leq 2\|x^{-1}\|^2 \|h\| \ .$$

Hint: $x + h = x(e + x^{-1}h)$.

Theorem *If A is a Banach algebra, then $G(A)$ is an open subset of A and the mapping $x \mapsto x^{-1}$ is a homeomorphism of $G(A)$ onto $G(A)$.*

Hint: Use the previous lemma.

Let $x \in A$. The *spectrum* $\sigma(x)$ of x is defined to be the set of all complex numbers λ such that $\lambda e - x$ is not invertible.

Examples: • For $\mathbf{x} = (x_1, \ldots, x_n) \in K^n$, $\sigma(x) = \{x_1, \ldots, x_n\}$.
- For $f \in C(S, K)$, $\sigma(f) = \{f(s) : s \in S\}$.
- For $A \in M_n(K)$, $\sigma(A)$ is the set of eigenvalues of A.

Theorem *If A is a complex Banach algebra and $x \in A$, then $\sigma(x)$ is compact and non-empty.*

Proof: Step 1: Show that $\lambda e - x$ is invertible for $|\lambda| > \|x\|$. Therefore $\sigma(x)$ is bounded. Also show that $g^{-1}\overline{(G(A))}$ is open where $g(\lambda) = \lambda e - x$ for $\lambda \in \mathbf{C}$. Hence $\sigma(x)$ is compact.
Step 2: Assume that $\sigma(x)$ is empty. Define $R : \mathbf{C} \to A$ by $R(\lambda) = (\lambda e - x)^{-1}$. If $h \in C$, $R(\lambda+h) - R(\lambda) = hR(\lambda + h)R(\lambda)$. (Hint: $a^{-1} - b^{-1} = a^{-1}(b - a)b^{-1}$.) Letting $h \to 0$ in $(T(R(\lambda + h)) - T(R(\lambda)))/h$, show that for any continuous linear functional $T : A \to \mathbf{C}$, $T \circ R$ is a holomorphic function on $\mathbf{C}$.
Step 3: For $|\lambda| > \|x\|$,

$$T \circ R(\lambda) = \sum_{n=0}^{\infty} \lambda^{-n-1} T(x^n) \; .$$

Step 4: Apply the Cauchy Theorem to the above identity and integrate term-by-term the right hand side to obtain $0 = T(e)$ for all T.

We now prove a version of the Hahn-Banach Theorem.

Lemma 3 *Let V be a Banach space, $W \subset V$, and f a linear functional on W satisfying $f(w) \leq \|w\|$ for all $w \in W$. Then f extends to a linear functional F on V satisfying $F(v) \leq \|v\|$ for all $v \in V$.*

Proof: Consider the family of extensions of f satisfying the required condition, i.e. (Y, f_Y) is in this family if $W \subset Y$ and f_Y restricted to W is f. This family can be partially ordered by inclusion. So by Zorn's Lemma has a maximal element (Z, f_Z).

Suppose Z is not equal to V. Let $v \in V \backslash Z$. f_Z can be extended to a linear functional on $Z + \mathbf{R}v$ as follows. For $w_1, w_2 \in Z$,

$$f(w_1) - \|w_1 - v\| \leq \|v + w_2\| - f(w_2) \; .$$

Let α be any number satisfying

$$\sup\{f(w) - \|w - v\| : w \in Z\} \leq \alpha \leq \inf\{\|v + w\| - f(w) : w \in Z\} \; .$$

Define $g : Z + \mathbf{R}v \to \mathbf{R}$ by $g(w + c \cdot v) = f(w) + c\alpha$. Then g is a linear functional extending f and satisfies $g(v) \leq \|v\|$. □

Corollary *Let F be as in the Lemma. Then F is a continuous linear functional on V.*

Hint: Show that $|F(v)| \leq \|v\|$ for all $v \in V$.

Step 5: Let W be the subspace generated by e and f defined on W by $f(c \cdot e) = c$. Apply the lemma above to reach a contradiction.

□

Theorem *(Gelfand-Mazur) If A is a complex Banach algebra in which every non-zero element is invertible, then A is isometrically isomorphic to the complex numbers.*

Proof: Show that $\sigma(x)$ consists of a single point $\lambda(x)$ and that $|\lambda(x)| = \|x\|$. □

Corollary *If A is a complex normed algebra in which every non-zero element is invertible, then A is isometrically isomorphic to the complex numbers.*