Project: The space of circles.

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1. We can map $\mathbb{R}^4$ to a set of curves in the plane by the map

$$\mathcal{C} : (a,p,q,r) \rightarrow \left\{(x,y) \in \mathbb{R}^2 : a\left(x^2+y^2\right) - 2px - 2qy + r = 0\right\}.$$

   We may then take a subset of the image curves to be circles. Let $\sim$ denote the equivalence relation $(a,p,q,r) \sim (\lambda a, \lambda p, \lambda q, \lambda r)$ for any $\lambda \neq 0$. Then the map above induces a map from $\mathbb{R}^4/\sim$. We consider this set the set of generalized circles in the plane. Find a subset of the domain so that the map is a homeomorphism to the space of (nondegenerate) circles in the plane. (Hint: define the topology on the space of curves so that this is easy.) Find a formula for the radius and center of $\mathcal{C}(a,p,q,r)$ if it represents a circle.

2. We can define a bilinear "dot" product on $\mathbb{R}^4$ in the following way

$$(a,p,q,r)\cdot(a',p',q',r') = \frac{1}{2}\left(2pp' + 2qq' - ra' - ar'\right).$$

   Recall that given an inner product $\langle\cdot,\cdot\rangle$ and a symmetric bilinear product $b(\cdot,\cdot)$ on a vector space V, a number λ is said to be an eigenvector of b with respect to $\langle\cdot,\cdot\rangle$ if there exists a vector $v \in V$ (said to be the corresponding eigenvector) such that $b(v,w) = \lambda\langle v,w\rangle$ for all $w \in V$. (Note that this can also be formulated in another way. The inner product gives an isomorphism between V and V^* given by $\phi : v \rightarrow \langle v,\cdot\rangle$. We also see that $b(v,\cdot) \in V^*$ and then map this transformation to V using the inner product. This gives a linear map. An eigenvalue of the bilinear product is the eigenvalue of that linear transformation.) What are the eigenvalues of this bilinear product with respect to the standard inner product on $\mathbb{R}^4$? Show that if $\mathcal{C}(a,p,q,r)$ and $\mathcal{C}(a',p',q',r')$ are intersecting circles,

$$(a,p,q,r)\cdot(a',p',q',r') = -aa'RR'\cos\phi$$

   where R and R' are the radii of $\mathcal{C}(a,p,q,r)$ and $\mathcal{C}(a',p',q',r')$ respectively and ϕ is the angle between the two circles. Give a condition for two circles to be exteriorly tangent. Describe the vectors with zero length, positive length, and negative length.

3. Prove Descartes' circle theorem: If four circles in the plane with radii R_1, R_2, R_3, R_4 are mutually tangent and their interiors are disjoint, then

$$\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}\right)^2 = 2\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}+\frac{1}{R_3^2}+\frac{1}{R_4^2}\right).$$

   Hint: Consider the generalized circle $\Theta = (0,0,0,1)$ which must be a linear combination of the four circles since the vector space is 4-dimensional. Then take dot products of the linear dependence relation with each of the circles and solve a linear system.

4. Describe the closure of the set of circles both in $\mathbb{R}^4/\sim$ and geometrically. (Hint: it should include lines.) Use this to generalize Descartes' circle theorem to the case when one circle is a line.

5. Consider the generalization to $(n-1)$-dimensional spheres in $\mathbb{R}^n$.