INTEGRATION WORKSHOP PROJECTS

1. Lie Brackets and parallel parking a car

$U \subseteq \mathbb{R}^n$ is an open set. A *smooth vector field* on U is a smooth mapping $v : U \to \mathbb{R}^n$.

A *smooth derivation* on $C^\infty(U)$ is a mapping $\delta : C^\infty(U) \to C^\infty(U)$ that satisfies the following properties:

$$\delta(f + \lambda g) = \delta(f) + \lambda\delta(g)$$
$$\delta(fg) = f\delta(g) + g\delta(f)$$

for all $f, g \in C^\infty(U), \lambda \in \mathbb{R}$.

(a) Show that there is a one-to-one correspondence between smooth vector fields and smooth derivations. (Hint: Consider $\delta_v(f) = v \cdot \nabla(f)$. What is the identification in the opposite direction?)

(b) If δ_1 and δ_2 are smooth derivations, show that their commutator, that is the map

$$f \mapsto \delta_1(\delta_2(f)) - \delta_2(\delta_1(f))$$

is also a smooth derivation.

(c) Given two smooth vector fields u, v, using the identification from part (a), and the result from (b), we can construct a new smooth vector field that corresponds to the commutator. This operation is called the *Lie Bracket* and is conventionally denoted by $[u, v]$, *i.e.*

$$\delta_{[u,v]}(f) = \delta_u(\delta_v(f)) - \delta_v(\delta_u(f))$$

Express the vector $[u, v]$ directly in terms of the vectors u and v.

(d) Associated with every vector field v is a dynamical system given by

$$\frac{dx}{dt} = v(x)$$

What is the interpretation for the dynamical system corresponding to the Lie bracket of two vector fields.

(optional) A simple model for the motion of a car is as follows:

1. The state of the car is described by a triple (x, y, θ) where $(x, y) \in \mathbb{R}^2$ give the location of the center of the car, and $\theta \in S^1$ is an angle that describes the orientation of the car.
2. For simplicity, we assume that the car can only turn at a fixed rate. The trajectory of the car in (x, y) is differentiable, and piecewise smooth. Each smooth piece being either a straight line along the orientation of the car (no turning), or circular arcs with a fixed radius a (turning to the right or the left at a constant rate). The trajectory $\theta(t)$ is piecewise linear, with $\theta' = 0$ for no turning, or $\theta' = \pm\beta$ otherwise.

Date: August 8, 2009.

(e) Determine the vector fields that correspond to the three possible motions of the car. Show that, it is indeed possible to parallel park a car with only these three allowed motions.

(f) Assuming that the car has only two allowed motions, namely turning to the left, or turning to the right with a fixed turning radius a, is it still possible to move in (roughly) a straight line in a given direction? Is it possible to parallel park this car?