Integration Workshop 2005

Project on the Stone-Weierstrass Theorem

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The goal of this project is to prove the Stone-Weierstrass Theorem. In 1885 Weierstrass proved that on a closed interval every polynomial can be uniformly approximated arbitrarily closely by polynomials. This result was generalized in 1937 by Stone.

Suppose that we have a collection of continuous functions on a compact Hausdorff space which is closed under addition, multiplication and scalar multiplication. Also suppose that for any two distinct points in the space, there is a continuous function in the collection which takes distinct values at these points. Then the theorem says that this collection approximates any continuous function arbitrarily closely.

Let X be a compact Hausdorff topological space. Let $C(X, \mathbf{R})$ (respectively, $C(X, \mathbf{C})$) denote the set of all continuous real-valued (respectively, complex-valued) functions on X. We provide $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$ with sup-norm metric. That is, for $f, g \in C(X, \mathbf{R})$ or $C(X, \mathbf{C})$, $d(f, g) = \|f - g\|_u = \sup_{x \in X} |f(x) - g(x)|$.

Let $\mathcal{A}$ be a subset of $C(X, \mathbf{R})$ (respectively, $C(X, \mathbf{C})$). $\mathcal{A}$ *separates points* if for every $x, y \in X$, $x \neq y$, there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. $\mathcal{A}$ is an *subalgebra* if $\mathcal{A}$ is a real (respectively, complex) vector subspace of $C(X, \mathbf{R})$ (respectively, $C(X, \mathbf{C})$) and $fg \in \mathcal{A}$ whenever $f, g \in \mathcal{A}$. $\mathcal{A} \subset C(X, \mathbf{R})$ is called a *lattice* if $\max(f, g)$ and $\min(f, g)$ are in $C(X, \mathbf{R})$ whenever f and g are.

Exercise: Show that $C(X, \mathbf{R})$ (and therefore $C(X, \mathbf{C})$) separates points.

Hint: Use the fact that a compact Hausdorff space is normal, hence Urysohn lemma holds.

Example: Let $X = \{x_1, ..., x_n\}$ with the discrete topology. Consider $h : C(X, \mathbf{R}) \to \mathbf{R}^n$ defined by $h(f) = (f(x_1), \ldots, f(x_n))$. Show that h is an algebra isomorphism if the multiplication in $\mathbf{R}^n$ is defined coordinate-wise.

The Stone-Weierstrass Theorem. *Let X be a compact Hausdorff topological space. If $\mathcal{A}$ is a closed subalgebra of $C(X, \mathbf{R})$ which separates points, then either $\mathcal{A} = C(X, \mathbf{R})$ or $\mathcal{A} = \{f \in C(X, \mathbf{R}) : f(x_0) = 0\}$ for some $x_0 \in X$. The first alternative is the case exactly when $\mathcal{A}$ contains all the constant functions in $C(X, \mathbf{R})$.*

We first prove several lemmas. The first lemma is the special case of the theorem for $X = \{x_1, x_2\}$.

Lemma. *The only subalgebras of $\mathbf{R}^2$ are $\mathbf{R}^2$, $\{(0,0)\}$, $\{(r,0) : r \in \mathbf{R}\}$, $\{(0,r) : r \in \mathbf{R}\}$, $\{(r,r) : r \in \mathbf{R}\}$.*

Hint: If a subalgebra $\mathcal{A}$ of $\mathbf{R}^2$ which contains $(a, b) \in \mathbf{R}^2$ with $a \neq b$, $a \neq 0$ and $b \neq 0$, then (a^2, b^2) is also in $\mathcal{A}$. Conclude that $\mathcal{A} = \mathbf{R}^2$ in this case. Determine what happens in the cases where there is no such element in $\mathcal{A}$.

We have to do a little calculus in preparation for the next lemma.

Lemma. *The Taylor's series of $f(t) = (1 - t)^{1/2}$ at 0 converges absolutely and uniformly to $f(t)$ on $[-1, 1]$.*

Proof: Step 1: Show that the Taylor's series of $f(t)$ converges absolutely and uniformly on $[-1, 1]$. Hint: Use the Stirling's formula:

$$\lim_{n \to \infty} \frac{n!}{\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}} = 1 \, .$$

Step 2: Let $g(t)$ be the limit of the Taylor's series for $t \in [-1, 1]$. Show that $2(1-t)g'(t) = -g(t)$ for $t \in (-1, 1)$. Solve this differential equation to conclude that $g(t) = f(t)$ for $t \in (-1, 1)$.

Step 3: Note that both f and g are continuous to finish the proofs. □

One of the consequences of Stone-Weierstrass Theorem will be that the polynomials are dense in $C([-1, 1], \mathbf{R})$ in the uniform norm. In the next lemma we prove that on $[-1, 1]$, $|x|$ is a limit of a sequence of polynomials which vanish at 0.

Lemma. *For any $\epsilon > 0$ there exists a polynomial with real coefficients such that $P(0) = 0$ and $||x| - P(x)| < \epsilon$ for all $x \in \mathbf{R}$.*

Proof: Step 1: Use the previous lemma to choose a polynomial $Q(x)$ such that $|(1-t)^{1/2} - Q(t)| < \epsilon/2$ for $t \in [-1, 1]$.

Step 2: Let $t = 1 - x^2$ and $R(x) = Q(1 - x^2)$ to get a polynomial $R(x)$ satisfying $||x| - R(x)| < \epsilon/2$ for $x \in [-1, 1]$.

Step 3: Finally, use $R(x)$ to construct a polynomial $P(x)$ such that $||x| - P(x)| < \epsilon$ for $x \in [-1, 1]$ and $P(0) = 0$. □

Now we prove that every closed subalgebra is a lattice:

Lemma. *If $\mathcal{A}$ is a closed subalgebra of $C(X, \mathbf{R})$, then $|f| \in C(X, \mathbf{R})$ for every $f \in C(X, \mathbf{R})$, and $\mathcal{A}$ is a lattice.*

Proof: Step 1: Let $\epsilon > 0$. For $0 \neq f \in C(X, \mathbf{R})$ let $h = f/\|f\|_u$, and use the previous lemma to obtain $\||h| - P \circ h\|_u < \epsilon$.

Step 2: Observe that $P \circ h \in \mathcal{A}$.

Step 3: Since $\mathcal{A}$ is closed and $\epsilon > 0$ is arbitrary, conclude that $|f| \in \mathcal{A}$. This finishes the proof of the first claim.

Step 4: Discover a way of expressing $\max(f, g)$ and $\min(f, g)$ in terms of f and g using the algebra operations and $|\cdot|$. Use this to show that $\mathcal{A}$ is a lattice. □

The last lemma says that if a closed lattice is sufficiently large, then it is quite large.

Lemma. *Let $\mathcal{A}$ be a closed lattice of $C(X, \mathbf{R})$. If $f \in C(X, \mathbf{R})$and for every $x, y \in X$ there exists $g_{xy} \in \mathcal{A}$ such that $g_{xy}(x) = f(x)$ and $g_{xy}(y) = f(y)$, then $f \in \mathcal{A}$.*

Proof: Step 1: Let $\epsilon > 0$. For each pair $x, y \in X$ let $U_{xy} = \{z \in X : f(z) < g_{xy}(z) + \epsilon\}$ and $V_{xy} = \{z \in X : f(z) > g_{xy}(z) - \epsilon\}$. Show that these sets open and contain x and y.

Step 2: Fix y. As x ranges over X, the sets U_{xy} cover X. Use compactness to find a finite subcover corresponding to a finite set of points, say, $x_1, \ldots, x_n \in X$. Let $g_y = \max(g_{x_1y}, g_{x_2y}, \ldots, g_{x_ny})$. Then $f < g_y + \epsilon$ on X and $f > g_y - \epsilon$ on the open set $V_y = \bigcap_{1 \leq i \leq n} V_{x_iy}$ which contains y.

Step 3: Now as y ranges over X, the sets V_y form an open cover of X. Get a finite subcover corresponding to the points, say, $y_1, y_2, \ldots, y_m \in X$ and let $g = \min(g_{y_1}, g_{y_2}, \ldots, g_{y_m})$. Then $||f - g||_u < \epsilon$.

Step 4: Use the fact that $\mathcal{A}$ is a closed lattice to finish the proof. □

We are now ready to prove the Stone-Weierstrass Theorem.

Proof of The Stone-Weierstrass Theorem: Step 1: For any pair of distinct points $x, y \in X$, let $\mathcal{A}_{xy} = \{(f(x), f(y)) : f \in \mathcal{A}\} \subset \mathbf{R}^2$. Observe that $\mathcal{A}_{xy}$ is a subalgebra of $\mathbf{R}^2$.

Step 2: Use The last two lemmas to conclude that $\mathcal{A} = C(X, \mathbf{R})$ if $\mathcal{A}_{xy} = \mathbf{R}^2$ for all x, y.

Step 3: If not, then there exist x, y such that $\mathcal{A}_{xy}$ is a proper subalgebra of $\mathbf{R}^2$. Use the first lemma to decide which subalgebra it can be. Conclude that there exists $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in \mathcal{A}$.

Step 4: Show that x_0 is unique since $\mathcal{A}$ separates points.

Step 5: Use The last two lemmas again to conclude that $\mathcal{A} = \{f \in C(X, \mathbf{R}) : f(x_0) = 0\}$.

Step 6: Finally observe that this can not be the case if $\mathcal{A}$ contains the constant functions. □

Corollary. *Let X be a compact subset of $\mathbf{R}^n$. Then the set of all polynomials is dense in $C(X, \mathbf{R})$.*

Remark: We want to prove a complex version of Stone-Weierstrass Theorem. But this would not be true without a further assumption: Consider the unit circle $X = \{z \in \mathbf{C} : |z| = 1\}$ in the complex plane $\mathbf{C}$. Then the polynomials in z with complex coefficients will separate points in X, but they will not be dense in $C(X, \mathbf{C})$. For instance, $f(z) = \bar{z}$ is not a limit of polynomials.

Exercise: Show that for $0 < \epsilon < 1$ there is no polynomial $P(z)$ such that $|\bar{z} - P(z)| < \epsilon$ for all $|z| = 1$.

Hint: Show that $\int_X zP(z)dz = 0$. Then compute $\int_X |z|^2 dz$ using $|z|^2 = z(\bar{z} - P(z)) + zP(z)$ to obtain a contradiction.

The Complex Stone-Weierstrass Theorem. *Let X be a compact Hausdorff topological space. If $\mathcal{A}$ is a closed subalgebra of $C(X, \mathbf{C})$ which separates points and is closed under complex conjugation, then either $\mathcal{A} = C(X, \mathbf{C})$ or $\mathcal{A} = \{f \in C(X, \mathbf{C}) : f(x_0) = 0\}$ for some $x_0 \in X$. The first alternative is the case exactly when $\mathcal{A}$ contains all the constant functions in $C(X, \mathbf{C})$.*

Hint: Apply the Stone-Weierstrass Theorem to the subalgebra $\mathcal{A}_{\mathbf{R}}$ of $C(X, \mathbf{R})$ consisting of all $(f + \bar{f})/2$ and $(f - \bar{f})/(2i)$ for $f \in \mathcal{A}$.