THE TYCHONOFF THEOREM[1]

The Tychonoff theorem asserts that the product of an arbitrary number of compact spaces is compact in the product topology. In Lecture Three we have proved this result for finitely many spaces, but unfortunately, the same method does not work for infinite products. The purpose of this project is to give a proof to the Tychonoff theorem in several steps.

First, one possible reformulation of the definition of compactness, which uses closed, rather than open sets is based upon the following notion:

Definition 1 *A collection $\mathcal{A}$ of subsets of a topological space X is said to have the* finite intersection property (FIP) *if the intersection of any finite subcollection is non-empty.*

Now we will state and prove an equivalent definition of compactness:

Theorem 2 *A topological space X is compact iff every collection of closed subsets of X with the FIP has a non-empty common intersection.*

Sketch of proof. Given a collection $\mathcal{A}$ of subsets of X, let $\mathcal{C}$ be the collection of their complements. Then $\mathcal{A}$ is a collection of open sets iff $\mathcal{C}$ is a collection of closed sets. The collection $\mathcal{A}$ is a covering of X iff the common intersection of subsets from $\mathcal{C}$ is empty. And, $\mathcal{A}$ has a finite subcovering iff there exist finitely many sets from $\mathcal{C}$ with empty intersection. Now, to prove the theorem, use the original definition of compactness and apply the contrapositive statement to the complements. ◯

The rough idea of our proof of the Tychonoff theorem is as follows. For a given collection of closed subsets $\mathcal{A}$ of the product space $\prod_\alpha X_\alpha$ with the FIP we will choose a largest collection $\mathcal{C} \supset \mathcal{A}$ of closed subsets with the FIP and then for each α consider the projection of this collection $\pi_\alpha(\mathcal{C})$ via the natural continuous projection maps $\pi_\alpha : X \to X_\alpha$. Then we will see that after taking the closures, this collection also has the FIP, and since X_α is compact, we can choose a point x_α in their common intersection. At last, we will prove that the point $\prod_\alpha x_\alpha$ lies in the common intersection of all the given closed sets.

Now we will proceed with the details and start with the following:

Lemma 3 *Let $\mathcal{A}$ be a collection of subsets of X, which has the FIP. Then there is a collection $\mathcal{D}$ of subsets of X such that $\mathcal{A} \subset \mathcal{D}$, $\mathcal{D}$ has the FIP, and there is no larger collection of subsets of X, that properly contains $\mathcal{D}$, with this property.*

Sketch of proof. The proof is based on Zorn's lemma, which says that any set A with

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strict partial ordering, in which every simply ordered subset has an upper bound, has a maximal element.

Let A be the set whose elements are all collections $\mathcal{B}$ of subsets of X such that $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B}$ has the FIP. The strict partial ordering on A is denoted by $\subset$. To prove the lemma, we must show that A has a maximal element $\mathcal{D}$.

Let B be a subset of A, that is simply ordered by $\subset$. Show that the collection

$$\mathcal{C} = \bigcup_{\mathcal{B} \in B} \mathcal{B}$$

is an element of A, which is the required upper bound on B in two steps: first, show that $\mathcal{A} \subset \mathcal{C}$ and second, that $\mathcal{C}$ has the FIP.

Finally, apply Zorn's lemma. ◯

Next statement is a rather straightforward, yet necessary step towards the main result.

Lemma 4 *Let $\mathcal{D}$ be a collection of subsets of X, which is maximal w.r.t. the FIP.*

(a) Any finite intersection of elements from $\mathcal{D}$ is an element of $\mathcal{D}$.

(b) If S is any subset of X, which intersects every subset from $\mathcal{D}$, then S is in $\mathcal{D}$ as well.

Proof. Fill in the details! ◯

Now we are ready for the final strike.

Theorem 5 (Tychonoff theorem) *Let*

$$X = \prod_{\alpha \in J} X_\alpha$$

be the product of compact topological spaces with the product topology. Then the space X is compact.

Sketch of proof. Start with a collection $\mathcal{A}$ of closed subsets of X with the FIP. Use Lemma 3 to choose a collection $\mathcal{D}$ containing $\mathcal{A}$, which is maximal w.r.t. the FIP. Determine that it would be sufficient to show that the common intersection of the closures of the subsets from $\mathcal{D}$ is nonempty:

$$\bigcap_{D \in \mathcal{D}} \bar{D} \neq \emptyset \quad .$$

Next, for a given $\alpha \in J$ show that the collection

$$\{\pi_\alpha(D), \text{ for } D \in \mathcal{D}\}$$

has the FIP and use the compactness of X_α to choose a point

$$\mathbf{x} = \prod_{\alpha \in J} x_\alpha \quad , \quad \text{where} \;\; x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)} \; .$$

Finally, one must show that $\mathbf{x} \in \bar{D}$ for every $D \in \mathcal{D}$. This is where we get to use the product topology and it is done in two steps.

First, show that if U_β is an open set in X_β containing $\pi_\beta(\mathbf{x})$, then $\pi_\beta^{-1}(U_\beta)$ intersects every element of $\mathcal{D}$.

And second, use Lemma 4 to show that every basis element of the product topology of X containing $\mathbf{x}$ belongs to $\mathcal{D}$.

Finally, use the FIP of $\mathcal{D}$ to see that every basis element containing $\mathbf{x}$ intersects every element of $\mathcal{D}$, and hence $\mathbf{x} \in \bar{D}$ for every $D \in \mathcal{D}$. ◯