INTEGRATION WORKSHOP PROJECT: THE URYSOHN METRIZATION THEOREM

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The Urysohn Metrization Theorem tells us under which conditions a topological space X is metrizable, i.e. when there exists a metric on the underlying set of X that induces the topology of X. The main idea is to impose such conditions on X that will make it possible to embed X into a metric space Y, by homeomorphically identifying X with a subspace of Y.

Let us start with some definitions. A T_1-space X (i.e. the space in which one-point sets are closed) is said to be *regular* if for any point $x \in X$ and any closed set $B \subset X$ not containing x, there exist two disjoint open sets containing x and B respectively. The space X is said to be *normal* if for any two disjoint closed sets B_1 and B_2 there exist two disjoint open sets containing B_1 and B_2 respectively.

Example. An example of a Hausdorff space which is not normal is given by the set $\mathbb{R}$, where the usual topology is enhanced by requiring that the set $\{1/n \mid n \in \mathbb{N}\}$ is closed. Examples of spaces which are regular but not normal exist, but are complicated.

Lemma. *Every regular space with a countable basis is normal.*

Proof. First, using regularity and countable basis, construct a countable covering $\{U_i\}$ of B_1 by open sets whose closures do not intersect B_2. Similarly, construct an open countable covering $\{V_i\}$ of B_2 disjoint from B_1. Then define

$$U'_n := U_n \setminus \bigcup_{i=1}^{n} \bar{V}_i \quad \text{and} \quad V'_n := V_n \setminus \bigcup_{i=1}^{n} \bar{U}_i.$$

Show that these sets are open and the the collection $\{U'_n\}$ covers B_1 and $\{V'_n\}$ covers B_2. Finally show that $U' := \cup U'_n$ and $V' := \cup V'_n$ are disjoint. ○

Next, we will prove one of the very deep basic results.

Urysohn lemma. *Let X be a normal space, and let A and B be disjoint closed subsets of X. There exists a continuous map $f : X \to [0,1]$ such that $f(x) = 0$ for every $x \in A$, and $f(x) = 1$ for every $x \in B$.*

Proof. Let Q be the set of rational numbers on the interval $[0,1]$. For each rational number q on this interval we will define an open set $U_q \subset X$ such that whenever $p < q$, we have $\bar{U}_p \subset U_q$. *Hint:* enumerate all the rational numbers on the interval (so that the

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first two elements are 1 and 0) and then define $U_1 = X \setminus B$ and all other U_q's can be defined inductively by using normality of X.

Now let us extend the definition of U_q to all rational numbers by defining $U_q = \emptyset$ if q is negative, and $U_q = X$ if $q > 1$.

Next, for each $x \in X$ define $Q(x)$ to be the set of those rational numbers such that the corresponding set U_q contains x. Show that $Q(x)$ is bounded below and define $f(x)$ as its infimum.

Now we will show that $f(x)$ is the desired function. First, show that if $x \in \bar{U}_r$, then $f(x) \leq r$, and if $x \notin U_r$, then $f(x) \geq r$.

Now prove the continuity of $f(x)$ by showing that for any $x_0 \in X$ and an open interval (c, d) containing $f(x_0)$, there exist a neighbourhood U of x_0 such that $f(U) \subset (c, d)$. [Why would this imply continuity?] For this choose two rational numbers q_1 and q_2 such that $c < q_1 < f(x_0) < q_2 < d$ and take $U = U_{q_2} \setminus \bar{U}_{q_1}$. ◯

Next, we will construct the metric space Y for the embedding. Actually, as a topological space, the space Y is simply the product of $\mathbb{N}$ copies of $\mathbb{R}$ with the product topology. Let $\bar{d}(a, b) = \min\{|a - b|, 1\}$ be the so-called standard bounded metric on $\mathbb{R}$ [show that this is indeed a metric]. Then if $\mathbf{x}$ and $\mathbf{y}$ are two points of Y, define

$$D(\mathbf{x}, \mathbf{y}) = \sup \left\{ \frac{\bar{d}(x_i, y_i)}{i} \right\}.$$

Show that this is indeed a metric.

Proposition. *The metric D induces the product topology on $Y = \mathbb{R}^{\mathbb{N}}$.*

Proof. First, let U be open in the metric topology and let $\mathbf{x} \in U$. We will find an open set V in the product topology such that $\mathbf{x} \subset V \subset U$. Choose an ε-ball centered at $\mathbf{x}$, which lies in U. Then choose N large enough so $1/N < \varepsilon$. Show that the following set satisfies the requirement:

$$V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_N - \varepsilon, x_N + \varepsilon) \times \mathbb{R} \times \mathbb{R} \times \cdots .$$

Conversely, consider a basis element $V = \prod_{i \in \mathbb{N}} V_i$ for the product topology, such that V_i is open in $\mathbb{R}$ and $V_i = \mathbb{R}$ for all but finitely many indices $i_1, ..., i_K$. Given $\mathbf{x} \in V$, we will find an open ball U in metric topology, which contains $\mathbf{x}$ and is contained in V. Choose an interval $(x_i - \varepsilon_i, x_i + \varepsilon_i)$ contained in V_i such that $\varepsilon_i < 1$ and define

$$\varepsilon = \min\{\varepsilon_i / i \mid i = i_1, ..., i_K\}.$$

Now show that the ball of radius ε centered at $\mathbf{x}$ is contained in V. ◯

Next we need the following technical result:

Lemma. *Let X be a regular space with a countable basis. There exists a countable collection of continuous functions $f_n : X \to [0, 1]$ such that for any $x_0 \in X$ and any* *neighbourhood U of x_0, there exists an index n such that $f_n(x_0) > 0$ and $f_n = 0$ outside U.*

Proof. Given x_0 and U, use regularity to choose two open sets B_n and B_m from the countable basis containing x_0 and contained in U such that $\bar{B}_n \subset B_m$. Then use the Urysohn lemma to construct a function $g_{n,m}$ such that $g_{n,m}(\bar{B}_n) = 1$ and $g_{n,m}(X \backslash B_m) = 0$. Now show that this collection of functions satisfies our requirement. ◯

Finally we will prove the main result:

Urysohn Metrization Theorem. *Every regular space X with a countable basis is metrizable.*

Proof. Given the collection of functions $\{f_n\}$ from the previous lemma, and $Y = \mathbb{R}^{\mathbb{N}}$ with the product topology, we define a map $F : X \to Y$ as follows:

$$F(x) = (f_1(x), f_2(x), ...).$$

Show that this is a continuous map. Also show that it is injective.

In order to finish the proof, we need to show that for each open set U in X, the set $F(U)$ is open in $F(X)$. Let z_0 be a point of $F(U)$. Let $x_0 \in U$ be such that $F(x_0) = z_0$ and choose an index N such that $f_N(x_0) > 0$ and $f_N(X \setminus U) = 0$. Now we let

$$W = \pi_N^{-1}((0, \infty)) \cap f(X),$$

where π_N is the projection $Y \to \mathbb{R}$ onto the Nth multiple. Show that W is an open subset of $F(X)$ such that $z_0 \in W \subset F(U)$. ◯

Give an example of a Hausdorff space with a countable basis which is not metrizable.