Topology Lectures – Integration workshop 2017

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Abstract

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1 Introduction to topology

1.1 Topology theorems

The basics of point set topology arise from trying to understand the following theorems from basic calculus: (in the following, we assume intervals written $[a, b]$ have the property $a < b$ so that they are not empty)

Theorem 1 (Intermediate Value Theorem) *If $f : [a, b] \to \mathbb{R}$ is a continuous function, and y is a number between $f(a)$ and $f(b)$ then there exists $x \in [a, b]$ such that $f(x) = y$.*

Theorem 2 (Extreme Value Theorem) *If $f : [a, b] \to \mathbb{R}$ is a continuous function, then there are numbers $x_m, x_M \in [a, b]$ such that*

$$f(x_m) = \min \{f(x) : x \in [a, b]\},$$
$$f(x_M) = \max \{f(x) : x \in [a, b]\}.$$

Here are some other interesting theorems in topology that we will not prove here:

Theorem 3 (Jordan Curve Theorem) *Any continuous simple closed curve in the plane, separates the plane into two disjoint regions, the inside and the outside.*

Theorem 4 (Jordan-Schoenfleiss Theorem) *For any simple closed curve in the plane, there is a homeomorphism of the plane which takes that curve into the standard circle.*

Theorem 5 (Generalized Jordan Curve Theorem) *Any embedding of the $(n-1)$-dimensional sphere into n-dimensional Euclidean space separates the Euclidean space into two disjoint regions.*

Theorem 6 (Brouwer Fixed Point Theorem) *Any continuous function from the closed disk in $\mathbb{R}^n$ to itself has a fixed point.*

Theorem 7 (Borsuk-Ulam Theorem) *If $f : S^2 \to \mathbb{R}^2$ is continuous, then there exists a point $x \in S^2$ such that $f(x) = f(-x)$.*

Theorem 8 (Invariance of Dimension) *No nonempty open subset of $\mathbb{R}^n$ is homeomorphic to an open subset of $\mathbb{R}^m$ if $m \neq n$.*

1.2 Topology of $\mathbb{R}^n$

For motivation, we recall what open and closed sets look like in $\mathbb{R}^n$. A set $U \subset \mathbb{R}^n$ is open if for any $x \in U$ there is a ball centered at x contained in U. A set F is closed if it contains all of its limit points, i.e., for every convergent sequence that is in F, the limit is in F. It can be shown that a set in $\mathbb{R}^n$ is closed if and only if its complement is open.

Let f be a function from $\mathbb{R}^n$ to $\mathbb{R}^k$ or from U to $\mathbb{R}^k$ where U is an open set in $\mathbb{R}^n$. Then the $\epsilon - \delta$ definition of continuity for f at x_0 is that $\forall \epsilon > 0, \exists \delta > 0$ such that $\|x - x_0\| < \delta$ implies $\|f(x) - f(x_0)\| < \epsilon$. (Here $\|\ \|$ denotes the usual distance function in $\mathbb{R}^n$ or $\mathbb{R}^k$.) If f is continuous at every point in its domain we say it is continuous. It then follows that f is continuous under this $\epsilon - \delta$ definition if and only if, for all open subsets U in $\mathbb{R}^k$, $f^{-1}(U)$ is open in $\mathbb{R}^n$.

The only structure of $\mathbb{R}^n$ that we need in the above is the ability to measure the distance betweeen two points in the space. So we can immediately generalize the above to a metric space. The above shows more, namely that we can do a lot if we just know what the open sets are, not the metric they came from. So we can abstract things by just looking at the collection of open sets. Note that the open sets in $\mathbb{R}^n$, and more generally the open sets in a metric space, have some obvious properties. The empty set and the whole space are open. Any union of open sets is an open set. Any finite intersection of open sets is open. These observations will be the basis for the definition of a topology.

1.3 Metric spaces

Definition 9 *A metric space (X, d) is a set X and a function (called the metric) $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$, the metric satisfies:*

1. *(positive definite) $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$*

2. *(symmetric) $d(x, y) = d(y, x)$*

3. *(triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$*

Definition 10 *In a metric space (X, d), a set U is said to be open if $\forall x \in U$, $\exists \epsilon > 0$ such that $d(y, x) < \epsilon$ implies $y \in U$.*

Proposition 11 *Let (X, d) be a metric space and let $\mathcal{T}$ be the collection of open sets in X. Then*

1. *$X \in \mathcal{T}$ and $\varnothing \in \mathcal{T}$,*

2. *Arbitrary unions of sets $U \in \mathcal{T}$ are in $\mathcal{T}$, i.e., for any indexing set I, if $U_i \in \mathcal{T}$ for all $i \in I$ then $\bigcup_{i \in I} U_i \in \mathcal{T}$,*

3. *If $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$.*

For any set X, we will define a collection of subsets $\mathcal{T}$ of X to be a **topology** for X if it satisfies the three properties above. Note that property 3 immediately implies by induction that a finite intersection of open sets produces an open set.

Definition 12 *A sequence x_n in a metric space (X, d) **converges** to a point $x \in X$ if $\forall \epsilon > 0$, there exists an index $N < \infty$ such that $n > N \implies d(x_n, x) < \epsilon$.*

Definition 13 *A subset F of a metric space (X, d) is **closed** if for every sequence x_n in F which converges to some x in X we have $x \in F$.*

Proposition 14 *A set F is closed if and only if $F^C = X \setminus F$ is open.*

Corollary 15 *Arbitrary intersections and finite unions of closed sets are closed.*

Definition 16 *The interior of a set A, denoted $\overset{\circ}{A}$ or $int(A)$, is*

$$int(A) = \{x : \exists \epsilon > 0 \quad s.t. \quad d(x, y) < \epsilon \implies y \in A\}$$

The closure of a set A, denoted $\bar{A}$ or $cl(A)$, is

$$cl(A) = \{x : \forall \epsilon > 0 \quad \exists y \in A \quad s.t. \quad d(x, y) < \epsilon\}$$

Proposition 17 *The interior of a set A is the union of all open sets contained in A. The closure of a set A is the intersection of all closed sets containing A.*

Note that the proposition shows that the interior is open since it is a union of open sets, and the closure is closed since it is an intersection of closed sets.

If the space X is a vector space, then one way to get a metric on X is to start with a norm.

Definition 18 *A function $|| \quad ||$ on X is a norm if*

- *For all $x \in X$, $||x|| \geq 0$, and $||x|| = 0$ if and only if $x = 0$.*

- *$||ax|| = |a| ||x||$ for every $a \in \mathbb{R}$ and $x \in X$.*

- $||x + y|| \leq ||x|| + ||y||$ *for every $x, y \in X$.*

A normed space is a vector space with a norm defined on it.

Proposition 19 *Let $(X, || \quad ||)$ be a normed space. Define $d(x, y) = ||x - y||$. Then (X, d) is a metric space.*

We end this section with some examples of metric spaces.

1. **Euclidean metric on $\mathbb{R}^n$:** The usual Euclidean norm gives a metric on $\mathbb{R}^n$.

$$d(x, y) = ||x - y|| = \left[\sum_{j=1}^{n} |x_j - y_j|^2 \right]^{1/2}$$

2. **$\mathbb{R}$ with a different topology:** Define a metric on X by

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}$$

   One of the homework problems is to check this is a metric and to determine if it gives a different topology for $\mathbb{R}$ from the standard one.

3. **Space of functions:** Let D be any set and let X be the set of all bounded real-valued functions on D. Define

$$d(f, g) = \sup_{x \in D} |f(x) - g(x)|$$

   Then (X, d) is a metric space.

4. **l^p norm on $\mathbb{R}^n$:** There are other norms we can put on $\mathbb{R}^n$ and hence other metrics. For $1 \leq p < \infty$, define

$$||x||_p = \left[\sum_{j=1}^{n} |x_j|^p \right]^{1/p}$$

   (The case $p = 2$ is the usual Euclidean metric.) It is not hard to show that we get the same collection of open sets, i.e, the same topology, for all the value of p. As $p \to \infty$ we get

5. **sup norm on $\mathbb{R}^n$:** The function

$$d(x, y) = \max_{1 \leq j \leq n} |x_j - y_j|$$

   is another metric on $\mathbb{R}^n$ that defines the same toplogy.

6. $l^p(\mathbb{N})$: If we look at infinite sequences instead of just vectors, things are more interesting. Let l^p be the set of sequences $(x_n)_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} |x_n|^p < \infty$. For such a sequence we define

$$||(x_n)_{n=1}^{\infty}||_p = \left[\sum_{n=1}^{\infty} |x_n|^p\right]^{1/p}$$

Consider the two sets

$$F = \{(x_n)_{n=1}^{\infty} \in l^p : x_n \geq 0 \,\forall n\}$$
$$U = \{(x_n)_{n=1}^{\infty} \in l^p : x_n > 0 \,\forall n\}$$

Is F closed? Is U open?

7. $l^{\infty}(\mathbb{N})$: The space is now the set of bounded infinite sequences. The norm is

$$||(x_n)_{n=1}^{\infty}||_{\infty} = \sup_{1 \leq n < \infty} |x_n|$$

1.4 Topological spaces

We define a topological space by specifying which sets are open. In order for this to be useful, we must put a few conditions on the collection of open sets.

Definition 20 *A topological space $(X, \mathcal{T})$ is a set X together with a collection $\mathcal{T}$ of subsets of X which satisfy:*

1. *$X \in \mathcal{T}$ and $\varnothing \in \mathcal{T}$,*

2. *Arbitrary unions of sets $U \in \mathcal{T}$ are in $\mathcal{T}$, i.e., for any indexing set I, if $U_i \in \mathcal{T}$ for all $i \in I$ then $\bigcup_{i \in I} U_i \in \mathcal{T}$,*

3. *If $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$.*

Instead of explicitly writing $U \in \mathcal{T}$, we usually say that U is open. Note that property 3 immediately implies by induction that a finite intersection of open sets produces an open set.

Definition 21 *A set F is closed if $F^C = X \setminus F \in \mathcal{T}$, i.e. if F^C is open.*

Proposition 22 *Arbitrary intersections and finite unions of closed sets are closed.*

Definition 23 *The interior of a set A, denoted $\overset{\circ}{A}$ or $int(A)$, is the union of all open sets contained in A. The closure of a set A, denoted $\bar{A}$ or $cl(A)$, is the intersection of all closed sets containing A.*

The interior is open since it is a union of open sets, and the closure is closed since it is an intersection of closed sets.

Proposition 24 *The interior of A is the largest open set contained in A. This means that $int(A)$ is open and if B is another open set contained in A, then $B \subset int(A)$. The closure of A is the smallest closed set containing A. This means that $\bar{A}$ is closed and if B is another closed set containing A, then $\bar{A} \subset B$.*

Definition 25 *A point $x \in X$ is a limit point of a set $A \subset X$ if every open set U containing x also contains a point $y \in A \setminus \{x\}$.*

We can characterize the closure in terms of limit points.

Proposition 26 *$\bar{A}$ is equal to the union of A and its limit points.*

Proof. Let F be a closed set containing A. Then $X \setminus F$ is an open set disjoint from A, so if x is a limit point of A it cannot be in $X \setminus F$, thus all limit points are contained in $\bar{A}$ (which is the intersection of all closed sets containing A). Conversely, if $x \in \bar{A} \setminus A$ then if there were an open set U containing x but disjoint from A, then $\bar{A} \cap U^C$ is closed set strictly contained in $\bar{A}$ containing A, a contradiction since $\bar{A}$ is the smallest such set. ■

Definition 27 *A sequence $x_n \in X$ converges to a point $y \in X$ if for all open sets $O \ni y$, there exists an index $N < \infty$ such that for all $n > N$, $x_n \in O$.*

The above definition is a direct generalization of the notion of convergence in metric spaces. However, there is no generalization of the notion of a Cauchy sequence to a topological space, since this requires the ability to compare the "size" of neighbourhoods at distinct points, and a topological structure does not allow for this comparison.

Definition 28 *$x \in A$ is an* isolated point *(of A) if there is an open set O such that $O \cap A = \{x\}$. $x \in X$ is an* accumulation point *of A if there exists a sequence in $A \setminus \{x\}$ that converges to x.*

Proposition 29 *In a metric space limit points and accumulation points are the same.*

However, in topological spaces limit points and accumulation points need not be the same. In general one should be cautious using sequences when working in a topological space. There are many characterizations of topogical properties in a metric space using sequences that do not carry over to topological spaces.

Of course all our previous examples of metric spaces are topological spaces. We end this section with a few trivial examples of topological spaces.
Example (discrete topology): We define all sets to be open. Does this topology come from a metric?

Example (coarse or indiscrete topology): The only open sets are X and $\varnothing$. Does this topology come from a metric?

Example (finite complement topology): A set is defined to be open if its complement is finite or the set is the empty set.

1.5 Continuous maps

We start with metric spaces.

Definition 30 *Let (X, d) and (Y, d') be metric spaces and $f : X \to Y$ a function. For $x \in X$, we say f is continuous at x if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \epsilon$. We say the map is globally continuous (or just continuous) if it is continuous at every point in X.*

We now give the definition for topological spaces.

Definition 31 *Let X and Y be topological spaces and $f : X \to Y$ a function. We say f is continuous if for any open set U in Y, $f^{-1}(U)$ is open.*

Of course metric spaces are topological spaces, so when X and Y are metric spaces we have two definitions of continuity.

Proposition 32 *If (X, d) and (Y, d') are metric spaces and $f : X \to Y$, then the above two definitions of continuity for f are equivalent.*

Note that for a continuous function the inverse image of a closed set is closed.

In a metric space we defined continuity at a point. We can do this in a general topological space as well. We define a *neighborhood* of a point x to be a set N containing x such that there is an open set U with $x \in U \subset N$. Note that an open set is a neighborhood of each of its members. Also note that a neighborhood does not have to be open. Now define a function $f : X \to Y$ from one topological space to another to be continuous at $x \in X$ if for every neighborhood V of $f(x)$, $f^{-1}(V)$ is a neighborhood of x. (Note that if V happens to be open, we are not asserting that $f^{-1}(V)$ is open.)

Continuous maps allow us to define equivalence of topological spaces.

Definition 33 *We say that two topological spaces are homeomorphic if there exists a continuous bijection between them with a continuous inverse. Such a map is called a homeomorphism.*

Example: One of the problems is to show that $\mathbb{R}$ and $(0, 1)$ are homeomophic. One of the problems in section three is to show that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.

Definition 34 *A function $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ is* sequentially continuous *if for every convergent sequence $x_n \to x$ in X, we have $f(x_n) \to f(x)$.*

Proposition 35 *Every continuous function is sequentially continuous. In a first countable space (for example, in a metric space), the converse is also true.*

1.6   Construction of topologies

Let X be a topological space and $Y \subset X$ be a subset. We can give Y the *subspace topology* by saying a set $U \subset Y$ is open if $U = V \cap Y$ for some open set $V \subset X$. It is easy to show that this gives a topology. Think about how this gives a topology on the sphere $S^n \subset \mathbb{R}^{n+1}$.

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, we say $\mathcal{T}_1$ is *finer* (or *stronger*) than $\mathcal{T}_2$ if $\mathcal{T}_2 \subset \mathcal{T}_1$. It is coarser or weaker if the inclusion goes the other way.

Proposition 36 *Let $\mathcal{S}$ be a collection of subsets of X. Then there is a unique topology $\mathcal{T}$ which is the weakest topology containing $\mathcal{S}$. This means that if $\mathcal{T}'$ is another topology containing $\mathcal{S}$, then $\mathcal{T}'$ is stronger than $\mathcal{T}$.*

Proof. Consider all the topologies that contain $\mathcal{S}$. (There is at least one - the discrete topology.) Define $\mathcal{T}$ to be their intersection. (Think carefully about what this means. The elements of a topology are subsets of X.) In other words, $\mathcal{T}$ is the collection of subsets U of X such that for every topology $\mathcal{T}'$ that contains $\mathcal{S}$ we have $U \in \mathcal{T}'$. It is now a matter of definition chasing to check that this works. ∎

We can think of the topology in the proposition as being formed by starting with $\mathcal{S}$ and adding "just enough" sets to get a topology.

Let X and Y be topological spaces. We can give $X \times Y$ a topology by taking the weakest topology that contains all sets of the form $U \times V$ where $U \subset X$ and $V \subset Y$ are open sets. (Note that not all open sets can be written as $U \times V$ for some $U \subset X$ and $V \subset Y$.) This construction is the *product topology*. The above immediately generalizes to a finite cartesian product.

We now have two ways to put a topology on $\mathbb{R}^n$. The metric topology we have already seen and the product topology that you get by thinking of $\mathbb{R}^n$ as the product of n copies of $\mathbb{R}$. Check that they are the same.

Proposition 37 *If Y is a set, $(X, \mathcal{T})$ is a topological space, and $f : X \to Y$ is a function, then we can define a topology $\mathcal{T}'$ on Y by taking $\mathcal{T}'$ to be all subsets U of Y such that $f^{-1}(U) \in \mathcal{T}$. This is the strongest topology on Y that makes f continuous.*

With this construction f is a continuous function. It is important to note that this construction works because of the set identities

$$f^{-1}(\cup_\alpha U_\alpha) = \cup_\alpha f^{-1}(U_\alpha)$$
$$f^{-1}(\cap_\alpha U_\alpha) = \cap_\alpha f^{-1}(U_\alpha) \tag{1}$$

Let X be a topological space and let $\sim$ be an equivalence relation. Recall that an equivalence relation $\sim$ is a relation satisfying the following properties:

1. (reflexivity) $x \sim x$.

2. (symmetry) $x \sim y$ implies $y \sim x$

3. (transitivity) $x \sim y$ and $y \sim z$ implies $x \sim z$.

Then $Q = X/\sim$ denotes the set of equivalence classes of the relation. For $x \in X$, we denote the equivalence class containing x by $[x]$. There is a natural quotient map $q : X \to Q$ given by $q(x) = [x]$. We now use the previous proposition to define a topology on the quotient space Q: the open sets in Q are the sets U such that $p^{-1}(U)$ is open in X. We call this the *quotient topology*

Example : The circle is a subset of the plane and so inherits a natural topology from the usual topology on the plane. Equivalently, the usual distance function on the circle is a metric which defines this topology. We can also think of the circle as the interval $[0, 2\pi]$ with the two endpoints identified. To be more precise, we define an equivalence relation by defining $0 \sim 2\pi$, and no other distinct points are equivalent. Since $[0, 2\pi]$ has a topology, we can consider the quotient topology on $[0, 2\pi]/\sim$. Show that $[0, 2\pi]/\sim$ is homeomorphic to the circle with the usual topology.

Given a map $f : X \to Y$ and a topology on X we have defined a topology on Y by taking advantage of the set identities (1). If instead we have a topology on Y, we might try to use it to define a topology by X by taking the collection of all sets of the form $f^{-1}(U)$ where U is open. We leave it to the reader to check that this does not work in general - the resulting collection of sets need not have the properties of a topology. We can define a toplogy by taking the weakest topology that includes all sets of the form $f^{-1}(U)$ where U is open in Y. With this definition f is a continuous function. In fact, this topology on X is weakest topology with this property.

We can generalize this construction. Suppose that the index α ranges over some index set $\mathcal{A}$ and for each α we have a topological space $(Y_\alpha, \mathcal{S}_\alpha)$ and a function $f_\alpha : X \to Y_\alpha$. Then we can define the induced (or *weak*) topology on X to be the weakest topology containing all sets of the form $f_\alpha^{-1}(U)$ where U is open in Y_α.

Proposition 38 *The* weak *topology constructed above is the weakest topology on X that makes all the functions f_α continuous.*

Remark: If $Y \subset X$ and X is a topological space, then Y inherits a natural topology (the subspace topology) from X. Another way to define this topology is that it is the weakest topology that make the inclusion map from Y to X continuous.

1.7   More exotic examples

- Line with two origins. We consider two copies of the real line. We denote elements of one of them by $(x, 1)$ where $x \in \mathbb{R}$ and the elements of the other by $(x, 2)$ where $x \in \mathbb{R}$. We define an equivalence relation by $(x, 1) \sim (x, 2)$ if $x \neq 0$. (Of course, all points are defined to be equivalent to themselves.) Note that $(0, 1)$ and $(0, 2)$ are not equivalent (hence the name). Their equivalence classes just contain one element. All other equivalence classes

contain two elements. The line with two origins is the quotient $\mathbb{R} \cup \mathbb{R}'/\sim$ where $x \sim x'$ if $x \neq 0$.

- Order topology. A total ordering on a set X is a relation $\leq$ such that for any $x, x' \in X$ we have either that $x \leq x'$ or $x' \leq x$ and both are true if and only if $x = x'$, and the relation is transitive. The order topology is the weakest topology that contains the "intervals"

$$(a, b) = \{x \in X : a < x \text{ and } x < b\}$$

Products of ordered sets can be given the dictionary order. What do you think the definition of the dictionary order is?

- Long line : This is a particular example of the previous example. Let $X = [0, 1) \times \mathbb{R}$. The "dictionary order" is a total order defined as follows. Given (x_1, y_1) and (x_2, y_2), to determine which is larger, we first look at the first component. If $x_1 < x_2$ we define $(x_1, y_1) < (x_2, y_2)$, and if $x_1 > x_2$ we define $(x_1, y_1) > (x_2, y_2)$. If $x_1 = x_2$ we look at the second coordinate. In this case if $y_1 < y_2$ we define $(x_1, y_1) < (x_2, y_2)$, and if $y_1 > y_2$ we define $(x_1, y_1) > (x_2, y_2)$. Then we use this total order to put the order topology on X. We can think of X as an uncountable number of copies of $[0, 1)$ glued together end to end.

- Zariski topology. Consider the following topology on $\mathbb{R}^n$. We take as the closed sets the sets

$$F(S) = \{x \in \mathbb{R}^n : f(x) = 0 \quad \forall f \in S\}$$

where S is a set of polynomials in n variables. Show that this is a topology on $\mathbb{R}^n$. Show that any two open sets must intersect, and hence the topology cannot be Hausdorff. (The definition of Hausdorff appears later.)

1.8   Local bases, basis, subbasis

Another way to specify a topology is with a local base (system of neighborhoods).

Definition 39 *Let X be a set, and for every $x \in X$, let there be given a collection $\mathcal{N}(x)$ of subsets of X satisfying*

1. *$V \in \mathcal{N}(x) \implies x \in V$.*

2. *If $V_1, V_2 \in \mathcal{N}(x)$, then $\exists V_3 \in \mathcal{N}(x)$ such that $V_3 \subseteq V_1 \cap V_2$.*

3. *If $V \in \mathcal{N}(x)$, then there exists a $W \in \mathcal{N}(x)$ such that $W \subset V$ and the following holds. If $y \in W$, then there exists $U \in \mathcal{N}(y)$ such that $U \subset V$.*

The collection $\{\mathcal{N}(x)|x \in X\}$ is a local base.
Given a local base, we can define a topology $\mathcal{T}$ by $O \in \mathcal{T}$ iff for all $x \in O$, there exists $V \in \mathcal{N}(x)$ such that $x \in V \subseteq O$.

Note that the neighborhoods of x in $\mathcal{N}(x)$ *do not have to be open*! However, given any local base, by "shrinking" the neighbourhoods a little if necessary, we can obtain a local base which generates the same topology, all of whose elements are open sets. In this case, condition 3 above simplifies to

$3'$. *If $V \in \mathcal{N}(x)$ and $y \in V$, then there exists $U \in \mathcal{N}(y)$ such that $U \subset V$.*

We can also specify a topology with a basis or a subbasis.

Definition 40 *A basis $\mathcal{B}$ is a collection of subsets of X such that*
(1) for all $x \in X$, there exists $U \in \mathcal{B}$ such that $x \in U$
(2) if $U, U' \in \mathcal{B}$ and $x \in U \cap U'$, then there is a set $U'' \in \mathcal{B}$ such that $x \in U''$ and $U'' \subset U \cap U'$.
A basis generates a topology by taking the open sets to be all sets we can form by taking a union of a collection of sets in $\mathcal{B}$. Equivalently we can define a set V to be open if every point $x \in V$ has a set $U \in \mathcal{B}$ such that $x \in U \subset V$.

An example of a basis is the open intervals for $\mathbb{R}$. Note that the basis determines the topology. The sets in the basis have to be open, but the basis itself need not be a topology since unions of elements of the basis are not necessarily in the basis.

We can also specify a topology through a subbasis.

Definition 41 *A subbasis $\mathcal{B}'$ (for a topology on X) is a collection of sets whose union is X. We define a topology by taking the open sets to be all sets which are the union of finite intersections of elements of B'.*

Given a subbasis $\mathcal{B}'$, define $\mathcal{B}$ to be all finite intersections of sets from $\mathcal{B}'$. Then $\mathcal{B}$ is a basis that generates the same topology as the subbasis.

Proposition 42 *Let X and Y be topological space. Then a basis for the product topology on $X \times Y$ is the collection of sets of the form $U \times V$ where U is an open set in X and V is an open set in Y.*

Proof. First we check that this collection of sets is a basis. Property 1 is immediate. For property 2, if $(x, y) \in (U \times V) \cap (U' \times V')$, then $(x, y) \in (U \cap U') \times (V \cap V')$ and $(U \cap U') \times (V \cap V')$ is in the basis. The product topology is the weakest topology containing the sets in the basis, and so coincides with the topology defined using the basis. ∎

1.9 Separation and countability

Here we simply list some of the separation and countability properties.
Separation:

- Hausdorff. A space is Hausdorff if for every two points $x, y \in X$, there are disjoint open sets U and V such that $x \in U$, $y \in V$. Note that a subspace of a Hausdorff space is Hausdorff but the quotient of a Hausdorff space may not be Hausdorff One of the problems is to show that the line with two origins is an example of this.

- Regular. A space is regular if one point sets are closed and for each pair of a point x and a closed set B disjoint from x there are disjoint open sets containing x and B.

- Normal. A space is normal if one point sets are closed and for each pair of disjoint closed sets A, B there are disjoint open sets containing A and B.

Hausdorff is the most important. One reason is the following.

Proposition 43 *Finite point sets in Hausdorff spaces are closed.*

Countability. A set is countable if there is a bijection between it and the natural numbers. It is easy to see that the integers, the even integers, and the rational numbers are all countable sets. It is also possible to see that the real numbers between 0 and 1 form an uncountable set using Cantor's diagonal argument. Topological spaces have the following countability axioms:

- First countable. A space is first countable if every point has a countable basis, i.e. given $x \in X$ there is a countable collection of open sets $U_1, U_2, U_3, \ldots$ such that for any neighborhood V of x, there is $k \in \mathbb{N}$ such that $U_k \subset V$.

- Second countable. A space is second countable if it has a countable basis for the topology. (Long line is an example which is not second countable.)

2 Compactness

2.1 Compactness in $\mathbb{R}^n$ and metric spaces

For a subset X of $\mathbb{R}^n$, the following three properties are equivalent. So we can take any one of them to be the definition of compact in $\mathbb{R}^n$.

- X is closed and bounded. (A subset of $\mathbb{R}^n$ is bounded if there is an $R > 0$ such that $||x|| \leq R$ for $x \in X$.)

- Every sequence contained in X has a limit point in X. That is, every sequence has a subsequence which converges to a point in X.

- Given any collection of open sets whose union contains X (an open cover of X), there is a finite subcollection whose union still contains X (finite subcover).

In a general topological space they are not equivalent. In a metric space the second and third are equivalent, but the first is not. We will first consider compactness in metric spaces and give a characterization of compactness in a metric space that is analogous to the first characterization of compact sets in $\mathbb{R}^n$.

We start by making the second and third properties above into definitions.

Definition 44 *A set F in a topological space $(X, \mathcal{T})$ is **compact** if for any collection of open sets whose union contains F (an open cover of F), there is a finite subcollection whose union still contains F (finite subcover).*

Definition 45 *A set F in a topological space $(X, \mathcal{T})$ is **sequentially compact** if every sequence contained in F has a limit point in F. That is, every sequence has a subsequence which converges to a point in F.*

Proposition 46 *In a metric space a set is compact if and only if it is sequentially compact.*

In a general topological space you can have sequentially compact sets which are not compact and compact sets which are not sequentially compact.

Remark: It is easily checked that a set $F \subset X$ is compact according to the above definition if and only if the space F with the relative topology is a compact topological space.

Definition 47 *A subset X of $\mathbb{R}^n$ is said to be* bounded *if there exists $r > 0$ such that $X \subset B(0, r) = \{x \in \mathbb{R}^n : |x| < r\}$.*

The Heine-Borel theorem says

Theorem 48 (Heine-Borel Theorem) *Subsets of $\mathbb{R}^n$ are compact if and only if they are closed and bounded.*

Proof. If a subset of $\mathbb{R}^n$ is compact, it must be closed since $\mathbb{R}^n$ is Hausdorff. The subset must be bounded because we can take the cover of $(-k, k)^n$ for $k = 1, 2, \ldots$ and it must have a finite subcover.

To prove the other direction, we start by showing that $[a, b]$ is compact in $\mathbb{R}$. For convenience we take $[a, b] = [0, 1]$. Let $\mathcal{U}$ be a cover of $[0, 1]$. We let

$$S = \{x \in [0, 1] : [0, x] \text{ has a finite subcover in } \mathcal{U}\}.$$

Now we show that $y = \sup S$ must be 1. Observe since $\mathcal{U}$ is a cover, y is contained in some open set $U \in \mathcal{U}$, and hence the interval $(y - \varepsilon, y + \varepsilon) \subset U$ for some small $\varepsilon > 0$. This implies both that $y \in S$ since there must be some $y' \in S$, $y' > y - \varepsilon$ since y is the sup, so take the finite cover of $[0, y']$ and add in U. But this also implies that $y + \varepsilon/2 \in S$ if $y + \varepsilon/2 \in [0, 1]$, so $y = 1$.

Since the topology on $\mathbb{R}^n$ is the same as the product topology it gets by thinking of it as the product of n copies of $\mathbb{R}$, $[a, b]^n$ is compact in $\mathbb{R}^n$. A closed and bounded set is a closed subset of some compact set $[-k, k]^n$, and thus is compact. ∎

In $\mathbb{R}^n$, compactness is equivalent to being closed and bounded, but this is not true in a general metric space. One of our examples is a metric space in which there are bounded closed sets that are not compact. (Which example?) There is a characterization of compactness in metric spaces. We have to replace boundedness by a stronger property and we also have to replace closed.

To see why we have to replace closed, consider the following example. Let Q be the rationals. Let $I = [0,1] \cap Q$. Then I is a closed set in Q. It is not sequentially compact and so is not compact. The problem in this example is that the space has "missing points."

Definition 49 *A sequence x_n in a metric space is Cauchy if for any $\epsilon > 0$ there exist an integer N such that for $n, m \geq N$ we have $d(x_n, x_m) < \epsilon$. A metric space X is complete if every Cauchy sequence converges, i.e., for every Cauchy sequence x_n there is $x \in X$ such that x_n converges to x.*

Definition 50 *In a metric space a set is totally bounded if for any $\epsilon > 0$, it can be covered by a finite number of balls of radius ϵ.*

Theorem 51 *A metric space is compact if and only if it is complete and totally bounded.*

2.2 Properties of compact spaces

In this section we see some properties of a compact set.

Proposition 52 *If F is compact and $A \subset F$ is closed, then A is compact.*
Proof. *Let U_α be an open cover of A. Since A is closed, A^c is open. Since $\cup_\alpha U_\alpha$ contains A, $A^c \cup (\cup_\alpha U_\alpha)$ contains F. (In fact it equals the entire space.) So this open cover of F admits a finite subcover. The finite subcover may or may not contain A^c, but if it doesn't we can add it to the finite subcover. So*

$$A \subset A^c \cup (\cup_{i=1}^n U_{\alpha_i}) \tag{2}$$

Since A^c does not cover any of A, this implies

$$A \subset \cup_{i=1}^n U_{\alpha_i} \tag{3}$$

so we have a finite subcover of A. ■

Proposition 53 *If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact.*
Proof. *If $\{U_i\}_{i \in I}$ is an open cover of $f(X)$, then $\{f^{-1}(U_i)\}_{i \in I}$ is an open cover of X. So it must have a finite subcover $\{f^{-1}(U_{i_j})\}_{j=1}^k$. But then $\{U_{i_j}\}_{j=1}^k$ must cover $f(X)$.* ■

Proposition 54 *If $f : X \to \mathbb{R}$ is continuous and X is compact, then there exist x_m and x_M in X such that*

$$f(x_m) = \inf_{x \in X} f(x)$$
$$f(x_M) = \sup_{x \in X} f(x).$$

In words, f attains its minimum and maximum.
Proof. *Since $f(X)$ is compact, it must be a closed and bounded subset of $\mathbb{R}$ and thus it contains its lower and upper bounds.* ■

Proposition 55 *If X and Y are compact topological spaces, then $X \times Y$ with the product topology is compact.*

Proof. Let $\mathcal{U}$ be an open cover of $X \times Y$. Define a subset A of X to be "good" if there is a finite subcover (from $\mathcal{U}$) for $A \times Y$. Our goal is to show X is good.

Consider an arbitary $x \in X$. We claim there is an open neighborhood V_x of x that is good. $\forall y \in Y$, (x, y) is a point in $X \times Y$ and so there is a $U_y \in \mathcal{U}$ with $(x, y) \in U_y$. Recall that a basis for the product is given by the products of an open set in X with an open set in Y. So there are open sets V_y in X and W_y in Y such that $(x, y) \in V_y \times W_y \subset U_y$. Now $\{W_y\}_{y \in Y}$ is a cover of Y and so has a finite subcover $\{W_{y_i} : i = 1, \cdots, n\}$. Define $V_x = V_{y_1} \cap \cdots \cap V_{y_n}$. Then V_x is an open neighborhood of x. $V_x \times Y$ is covered by $\cup_{i=1}^{n}(V_{y_i} \times W_{y_i})$ which is itself covered by $\cup_{i=1}^{n} U_{y_i}$, i.e., finite subcover of $\mathcal{U}$. So V_x is good.

Now $\forall x \in X$, let V_x be a good open neighborhood of x. Then V_x is a cover of X and so has a finite subcover, i.e.,

$$X \subset \cup_{i=1}^{n} V_{x_i}$$

Each $V_{x_i} \times Y$ has a finite subcover from $\mathcal{U}$, and the union of these finite subcovers will be a finite subcover of $X \times Y$. ∎

Proposition 56 *Compact subsets of a Hausdorff space are closed.*

2.3 Examples and non-examples of compact spaces

- Any finite topological space is compact.

- The finite-dimensional sphere $\left\{ x \in \mathbb{R}^n : |x|^2 = 1 \right\}$ is compact.

- The Cantor set is compact.

- For any space with the finite-complement topology, every subset is compact. The is a reflection of the fact that there are not very many open sets in this topology. (The stronger the topology, the harder it is for a set to be compact.)

- Let l^2 be the set of sequences $(x_n)_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} x_n^2 < \infty$ and

$$||(x_n)_{n=1}^{\infty}|| = \left[\sum_{n=1}^{\infty} x_n^2\right]^{1/2} \tag{4}$$

  The unit ball is this space is a closed and bounded set, but it is not compact. In fact, it is not sequentially compact. For example, let e_n be the sequence which is 1 in the nth place and 0 elsewhere. Then $d(e_n, e_m) = \sqrt{2}$ for $n \neq m$, so e_n cannot have a convergent subsequence. One of the exercises is to prove that in this space every compact set has empty interior. Another exercise is to prove that the following set is compact:

$$\{(x_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} n x_n^2 \leq 1\} \tag{5}$$

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- Let $L^2(\mathbb{R})$ be the space of functions f on $\mathbb{R}^2$ with $\int f^2(x)dx < \infty$. We define

$$||f||_2 = \left[\int f^2(x)\,dx \right]^{1/2} \tag{6}$$

  (There are some major issues here which we ignore.) The closed unit ball is not compact in this space. (Can you prove this?) For $g \in L^2$, define

$$T_g(f) = \int f(x)g(x)\,dx \tag{7}$$

  Look at the weakest topology that makes all these functions T_g (where g ranges over L^2) continuous. A big theorem from functional analysis says the closed unit ball is compact in this topology.

3 Connectedness

3.1 Connected and disconnected sets in $\mathbb{R}^n$

The key property of a connected set is the intermediate value theorem, which states that if $f : [a,b] \to \mathbb{R}$ is a continuous function and $f(a) \le r \le f(b)$ then there exists $c \in [a,b]$ such that $f(c) = r$. Notice this is not true for functions on disconnected sets such as $(0,1) \cup (1,2)$.

3.2 Definition of connected

Definition 57 *A separation of a space X is a pair U, V of disjoint open subsets of X such that $X = U \cup V$. Note that the two sets U and V are both open and closed since $U = X \setminus V$ and $V = X \setminus U$. The trivial separation consists of X and $\varnothing$.*

Definition 58 *A space X is* connected *if there exist no nontrivial separations of X. Equivalently, X is connected if the only open and closed subsets of X are X and $\varnothing$ (since if $A \subset X$ is open and closed, then $X = A \cup A^C$ is a separation if neither is empty). A space which is not connected is said to be* disconnected. *A subset of a topological space is connected if it is connected as a toplogical space itself when we endow it with the subspace topology.*

Example 59 $(0,1)$ *is connected.*

Example 60 $(0,2) \setminus \{1\}$ *is disconnected since $(0,1)$ and $(1,2)$ form a nontrivial separation.*

Using the definition of the subspace topology, a subset A of X is not connected if we can find open sets U and V in X such that $A \subset U \cup V$, $A \cap U \ne \emptyset$ $A \cap V \ne \emptyset$ and $U \cap V \cap A = \emptyset$. One might ask if it is always possible to choose these sets so that $U \cap V = \emptyset$. It is not.

Example 61 *Let* $X = \{a, b, c\}$. *Let*

$$\mathcal{T} = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\} \tag{8}$$

and let $S = \{a, c\}$. *The subspace topology on* S *is*

$$\mathcal{T}' = \{\emptyset, \{a\}, \{c\}, \{a, c\}\} \tag{9}$$

i.e., the discrete topology. So $S = \{a\} \cup \{c\}$ *is a separation that shows* S *is not connected. But there are no disjoint open sets* U, V *in* X *with* $U \cap S = \{a\}$ *and* $V \cap S = \{c\}$.

A note on the proof of Heine-Borel: we essentially used that $[0, 1]$ is connected and showed that the set of points $y \in [0, 1]$ such that $[0, y]$ can be covered by a finite subcover is both open and closed, and hence must be everything.

3.3 Properties of connected sets

Proposition 62 *The union of a collection of connected sets whose intersection is not empty is a connected set.*

Proof. Let Y be the topological space. Let $X_i \subset Y$ be connected. Let $X = \cup_i X_i$. We give X the subspace topogy. Note that there are two ways to put a topology on X_i, the subspace toplogy we get by thinking of it as a subset of Y and the subspace toplogy we get by thinking of it as a subset of X. We leave it to the reader to check they are the same. So the original assumption that X_i is connected as a subset of Y means it is connected as a subset of X.

Let $x \in \cap_i X_i$. Now suppose $X = U \cup V$ where U and V are disjoint open subsets of X. x must belong to one of U and V. Assume it belongs to U. Now $X_i \cap U$ and $X_i \cap V$ are open sets in X_i in the subspace topology and $X_i \cap U$ is not empty since it contains x. So $X_i \cap V$ must be empty, i.e., $X_i \subset U$. This is true for all i, so $X = U$, and so V is empty. ■

Proposition 63 *Let* A *be a connected subset of* X. *If* $A \subset B \subset \bar{A}$ *then* B *is connected.*

Proof. Suppose $B = U \cup V$, where U and V are disjoint and open in the subspace topology for B. We leave it to the reader to check that $U \cap A$ and $V \cap A$ are open in A with the subspace topology. They are clearly disjoint and cover A. Since A is connected, one of them must be empty. Assume that $V \cap A$ is empty. So A is entirely contained in U.

Since U, V are open in B, there are open sets U', V' in X with $U = B \cap U'$ and $V = B \cap V'$ Since $A \subset U$, $A \subset (V')^c$. Since $(V')^c$ is closed in X, $\bar{A} \subset (V')^c$. This implies $B \subset (V')^c$. If $x \in V$, then $x \in B$, and so $x \notin V'$. But $V \subset V'$, a contradiction. So V is empty. This shows B is connected. ■

Proposition 64 *The product of connected sets is connected.*

Proof. We see that $\{x\} \times Y$ and $X \times \{y\}$ are connected. Since both contain (x, y), $V_{x,y} = (\{x\} \times Y) \cup (X \times \{y\})$ is connected. Now pick some $y_0 \in Y$. We see that

$$\bigcup_{x \in X} V_{x,y_0} = X \times Y$$

and

$$\bigcap_{x \in X} V_{x,y_0} = X \times \{y_0\} \neq \varnothing.$$

Thus $X \times Y$ is connected. ■

Proposition 65 *If $f : X \to Y$ is continuous and X is connected then $f(X)$ is connected.*

Proof. Exercise ■

Proposition 66 *(Intermediate value theorem) If X is connected, $f : X \to \mathbb{R}$ is continuous, and $f(a) \leq r \leq f(b)$ then there exists $c \in X$ such that $f(c) = r$.*

Proof. We know that $f(X)$ is a connected subset of $\mathbb{R}$. Now if there is no c such that $f(c) = r$, then we can cover $f(X)$ by the sets $(-\infty, r) \cap f(X)$ and $(r, \infty) \cap f(X)$, which are disjoint open sets. They are nonempty since one contains $f(a)$ and the other $f(b)$. This is a separation, contradicting that $f(X)$ is connected. ■

3.4 Path connected

Definition 67 *A path in X is a continuous map $\gamma : [a, b] \to X$.*

Definition 68 *A space X is* path connected *if any two points can be joined by a path.*

One of the problems gives an example of a set that is connected but not path connected. So these two notions are not equivalent. However, one is stronger than the other.

Proposition 69 *If X is path connected, then it is connected.*

Proof. We shall show that if X is not connected, then it is not path connected. If X is not connected, then there is a separation $\{U, V\}$. Given $x, y \in X$ if there were a path $\gamma : [a, b] \to X$ between them, then $\gamma([a, b])$ would be connected, which implies the path must lie entirely in U or V (otherwise U and V would form a separation for $\gamma([a, b])$), which says that there are no paths between points in U and points in V. Hence X is not path connected. ■

3.5   Components

Definition 70 *Given X, we can define an equivalence relation on X by setting $x \sim y$ if there is a connected subset containing both x and y. The equivalence classes are called* components *or* connected components *of X.*

Show that this is an equivalence relation.

Proposition 71 *The components of X are connected disjoint subsets of X whose union is X, such that each connected subset of X intersects only one component.*

Proof. Let $\{C_i\}_{i \in I}$ be the components. Since the components are equivalence classes, they must be disjoint and must cover. If U is connected and $x_i \in U \cap C_i$ and $x_j \in U \cap C_j$ then $x_i \sim x_j$, which implies that $C_i = C_j$ by the definition of components. Now we must show that components are connected. Fix $x_0 \in C_i$. For any $x \in C_i$, there is a connected set A_x containing both x_0 and x since $x \sim x_0$. Thus $C_i = \bigcup_{x \sim x_0} A_x$, which implies that C_i is connected since it is the union of connected sets with a common intersection point x_0. ∎

We can also look at path components.

Definition 72 *Define an equivalence relation on X by $x \sim y$ if there is a path from x to y. The equivalence classes are called* path components *of X.*