Topology problems

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1 Problems on topology

1.1 Basic Problems

1-1. Applications of theorems

(a) Use the Intermediate Value Theorem to show that there is a number $c \in [0, \infty)$ such that $c^2 = 2$. We call this number $c = \sqrt{2}$.
Answer: Since the function $f(x) = x^2$ is continuous and $f(1) = 1$ and $f(2) = 4$, there must be a number c between 1 and 2.

(b) Use the Extreme Value Theorem to show Rolle's theorem: If $f : [a, b] \to \mathbb{R}$ is differentiable and $f(a) = f(b)$ then there is a $c \in [a, b]$ such that $f'(c) = 0$.

(c) Use the Jordan Curve Theorem to show there is no continuous injective map of the complete graph K_5 into the plane.

1-2. In this problem we use our analysis definitions of topology on $\mathbb{R}$ and confirm the topological versions. An open set is in $\mathbb{R}$ a set U such that for any $x \in U$ there is an $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq U$. A closed set is a set which contains all of its limit points, i.e. if $x_i \in F$ for all i, then $\lim_{i\to\infty} x_i \in F$.

(a) Show that every open subset of $\mathbb{R}$ is the complement of a closed set and every closed subset is the complement of an open set.
Answer: Suppose U is open and x_i is a sequence in $\mathbb{R} \setminus U$ with $\lim x_i = x_\infty$. If $x_\infty \in U$ then $(x_\infty - \varepsilon, x_\infty + \varepsilon) \subseteq U$, but that means that $x_i \in U$ for i large enough, which is not true. This means that x_∞ must be in $\mathbb{R} \setminus U$ and it is closed. Suppose $\mathbb{R} \setminus F$ is not open. Then there exists a point $x \in \mathbb{R} \setminus F$ and a sequence $x_i \in F$ converging to $x \notin F$ (since $\left(x - \frac{1}{n}, x + \frac{1}{n}\right) \subsetneq \mathbb{R} \setminus F$), which means that F is not closed.

(b) We say a function $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}$, is continuous if for any sequence $\{x_i\} \subseteq D$ with $\lim_{i\to\infty} x_i \in D$,
$$\lim_{i\to\infty} f(x_i) = f\left(\lim_{i\to\infty} x_i\right).$$
Show that if $F \subseteq \mathbb{R}$ is closed, then $f^{-1}(F)$ is closed.

(c) Show that if $U \subseteq \mathbb{R}$ is open, then $f^{-1}(U)$ is open.

(d) Show that every open set is the union of intervals.

(e) If $U \subseteq \mathbb{R}$ is open, show that a function $f : U \to \mathbb{R}$ is continuous (meaning the preimage of an open set is open) if and only if for every $x \in U$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ if $|x - y| < \delta$.

1-3. **(closed sets) A set F is *closed* if $F^C = X \setminus F \in \mathcal{T}$, i.e. if F^C is open.

(a) Show that arbitrary intersections and finite unions of closed sets are closed.

(b) Show that a map $f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is closed for any closed set $V \subseteq Y$.

(c) A point $x \in X$ is a *limit point* of a set $A \subset X$ if every open set U containing x also contains a point $y \in A \setminus \{x\}$. The *closure* of A, denoted $cl(A)$ or $\bar{A}$, is the intersection of all closed sets containing A. Show that $\bar{A}$ is closed.

(d) Show that $\bar{A}$ is equal to the union of A and its limit points.

(e) Show that if A is closed, then $\bar{A} = A$ and A contains its limit points.

(f) (accumulation points) A point $x \in X$ is an *accumulation point* of A if there exists a sequence in $A \setminus \{x\}$ that converges to x. A point $x \in A$ is an *isolated point* (of A) if there is an open set O such that $O \cap A = \{x\}$. Let $A \subseteq \mathbb{R}$, and A' denotes the set of all the accumulation points of A. If $y \in A'$ and $U \subseteq \mathbb{R}$ is an open set containing y, show that there are infinitely many distinct points in $A \cap U$.

(g) Let $A \subseteq \mathbb{R}$, and A' denotes the set of all the accumulation points of A. If $y \in A'$ and $U \subseteq \mathbb{R}$ is an open set containing y, show that there are infinitely many distinct points in $A \cap U$.

(h) Let $A \subseteq \mathbb{R}$, show that

$$A' = \bigcap_{x \in A} \overline{A \backslash \{x\}}.$$

(i) Give an example of a limit point that is not an accumulation point in a topological space.

1-4. (interior) The interior of A, denoted $int(A)$ or $\overset{\circ}{A}$ is the union of all open sets contained in A.

(a) Show $int(A)$ is open.

(b) Show that if B is an open set contained in A, then $B \subseteq int(A)$.

1.2 Examples of topologies

1-5. **(metric topology) A *metric space* (X, d) is a set X and a function (called the metric) $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$, the metric satisfies:

- (positive definite) $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$
- (symmetric) $d(x, y) = d(y, x)$
- (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$

The ball of radius $r > 0$ centered at $x \in X$ is defined to be $B(x, r) = \{y \in X : d(x, y) < r\}$. We can generalize the definition of open set on $\mathbb{R}$ to open set on a metric space by saying a set $U \subseteq X$ is open if for every $x \in U$ there exists an $r > 0$ such that $B(x, r) \subseteq U$.

(a) Show that $d(x, y) = |x - y|$ makes $\mathbb{R}$ into a metric space.

(b) Show that the open sets described above form a defines a topology. This is called the *metric topology*.

(c) Show that a function $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is continuous if and only if for every $x \in X$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ if $d_X(x, y) < \delta$.

(d) Show that in a metric space limit points and accumulation points are the same.

1-6. (Norms on vector spaces) Let V be a vector space. A function $\|\cdot\| : V \to \mathbb{R}$ on V is a *norm* if

- For all $x \in V$, $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$.
- $\|ax\| = |a| \|x\|$ for every $a \in \mathbb{R}$ and $x \in V$.
- $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in V$.

(a) Show that $d(x, y) = \|x - y\|$ defines a metric.

(b) Let $C((0,1))$ be the set of bounded continuous functions on $(0,1)$ and show that $\|f\| = \sup_{x\in(0,1)} |f(x)|$ is a norm. Let $U = \{f : f(x) > 0 \quad \forall x \in (0,1)\}$, and $V = \{f : f(x) \geq 0 \quad \forall x \in (0,1)\}$. For each of U and V determine if the set is open or closed or neither. You should prove your answer.

(c) Let l^p be the set of sequences $(x_n)_{n=1}^\infty$ with $\sum_{n=1}^\infty |x_n|^p < \infty$. For such a sequence we define

$$||(x_n)_{n=1}^\infty||_p = \left[\sum_{n=1}^\infty |x_n|^p\right]^{1/p}.$$

Show this is a norm. Consider the two sets

$$F = \{(x_n)_{n=1}^\infty \in l^p : x_n \geq 0 \, \forall n\}$$
$$U = \{(x_n)_{n=1}^\infty \in l^p : x_n > 0 \, \forall n\}$$

Is F closed? Is U open?

1-7. For $x, y \in \mathbb{R}$ let

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}$$

(a) Show this is a metric.

(b) Does this metric give $\mathbb{R}$ a different topology from the one that comes from the usual metric on $\mathbb{R}$? You should prove your answer.

(c) Generalize this to show that for any metric space (X, d), there is a bounded metric that generates the same topology.

1-8. (local base) Let X be a set. A *local base (or basis)*, or *neighborhood base*, is a collection $\{\mathcal{N}(x) | x \in X\}$ of subsets of X satisfying

- $V \in \mathcal{N}(x) \implies x \in V$.
- If $V_1, V_2 \in \mathcal{N}(x)$, then $\exists V_3 \in \mathcal{N}(x)$ such that $V_3 \subseteq V_1 \cap V_2$.
- If $V \in \mathcal{N}(x)$, then there exists a $W \in \mathcal{N}(x)$ such that $W \subseteq V$ and the following holds: If $y \in W$, then there exists $U \in \mathcal{N}(y)$ such that $U \subseteq W$.

(a) Given a local base $\{\mathcal{N}(x)\}$, define a set $\mathcal{T}$ by $U \in \mathcal{T}$ if and only if for any $x \in U$, there exists $V \in \mathcal{N}(x)$ such that $V \subseteq U$. Show that $\mathcal{T}$ is a topology. We call $\{\mathcal{N}(x)\}$ a local base of the topology.

(b) Suppose $\{\mathcal{N}(x)\}$ is a local base for a topological space $(X, \mathcal{T})$. If $A \in \mathcal{N}(x)$, show that there exists a $U \in \mathcal{T}$ such that $x \in U \subseteq A$. Hence neighborhoods need not be open, but must contain an open set.

(c) Show that open balls $B_r(x) = \{y : d(x,y) < r\}$ form a local base of a metric space (X, d).

(d) Show that closed disks $D_r(x) = \{y : d(x,y) \leq r\}$ (where $r > 0$) form a local base of a metric space (X, d).

(e) $\mathcal{N}_1(x)$ and $\mathcal{N}_2(x)$ are local bases for a space X. Show that the topology $\mathcal{T}_1$ generated by $\mathcal{N}_1(x)$ is finer that the topology $\mathcal{T}_2$ generated by $\mathcal{N}_2(x)$ if and only if for all $B \in \mathcal{N}_2(x)$, there is a set $A \in \mathcal{N}_1(x)$ such that $x \in A \subseteq B$.

1-9. (finer/coarser) In this problem and the next, we explicitly write the topology as a collection of open sets $\mathcal{T}$. Suppose $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are topological spaces and $\mathcal{T}_2 \subseteq \mathcal{T}_1$. We say the topology $\mathcal{T}_1$ is *finer* than $\mathcal{T}_2$ and the topology $\mathcal{T}_2$ is *coarser* than $\mathcal{T}_1$.

(a) Show that $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ if and only if for every $x \in X$ and every $U \in \mathcal{T}_2$ with $x \in U$, there is a $V \in \mathcal{T}_1$ with $x \in V$ such that $V \subseteq U$.

(b) Show that $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ if and only if the identity map $\mathrm{Id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous.

1-10. **(basis for a topology) A *basis* $\mathcal{B}$ is a collection of subsets of X such that

- For all $x \in X$, there exists $U \in \mathcal{B}$ such that $x \in U$.
- If $U, U' \in \mathcal{B}$ and $x \in U \cap U'$, then there is a set $U'' \in \mathcal{B}$ such that $x \in U''$ and $U'' \subset U \cap U'$.

(a) Show that a basis generates a topology by taking the open sets to be all sets we can form by taking a union of a collection of sets in $\mathcal{B}$.

(b) Show this is equivalent to defining a set V to be open if every point $x \in V$ has a set $U \in \mathcal{B}$ such that $x \in U \subseteq V$.

(c) Show that the collection of all balls forms a basis for the metric topology. Show that not every open set is in the basis.

1-11. **(Product topology) Let X_α be a collection of topological spaces indexed by a set $\mathcal{I}$. We define a topology on the Cartesian product $\prod_{\alpha \in \mathcal{I}} X_\alpha$ as follows: a basis is given by sets of the form $\prod_{\alpha \in \mathcal{I}} U_\alpha$, where $U_\alpha \subseteq X_\alpha$ is open and for all but finitely many α , $U_\alpha = X$.

(a) Prove that the above construction does indeed yield a basis.

(b) Prove that this is the coarsest topology (with fewest open sets) such that the projections $\pi_\alpha : \prod_{\alpha \in \mathcal{I}} X_\alpha \to X_\alpha$ are continuous.

1-12. **(Subspace topology) Suppose $(X, \mathcal{T})$ is a topological space.

(a) If $W \subseteq X$ is a subset, show that the *relative* or *subspace* topology of W defined by $O \subseteq W$ is open only if $O = W \bigcap U$ for some $U \in \mathcal{T}$ is indeed a topology.

(b) Let $S_2 \subset S_1 \subset X$. Equip S_1 with the subspace topology. There are two ways to define a topology on S_2. We can give it the subspace topology it gets by thinking of it as a subset of X or we can give it the subspace topology it gets by thinking of it as a subset of S_1. Show that there two topologies on S_2 are the same.

1-13. (Initial topology) Given a map $f : X \to Y$ and a topology on Y, let $\mathcal{T}$ be the collection of subsets of X of the form $f^{-1}(U)$ where U is open.

(a) Show that this defines a topology on X. This is called the *initial topology* or the *weak topology.*

(b) Suppose that the index α ranges over some index set $\mathcal{A}$ and for each α we have a topological space $(Y_\alpha, \mathcal{S}_\alpha)$ and a function $f_\alpha : X \to Y_\alpha$. In this case, we cannot simply take the sets of the form $f_\alpha^{-1}(U)$ where U is open. Instead, we need to take the coarsest topology (fewest open sets) containing this set. We call this initial topology (or weak topology) as well. Show that the initial topology constructed above is the weakest topology on X that makes all the functions f_α continuous.

(c) Show that the subspace topology is the initial topology for the inclusion map.

(d) Show that the product topology is the initial topology for the projection maps π_α.

1-14. **(quotient topology) Let X be a topological space and let $\sim$ be an equivalence relation. Recall that an equivalence relation $\sim$ is a relation satisfying the following properties:

- (reflexivity) $x \sim x$.
- (symmetry) $x \sim y$ implies $y \sim x$
- (transitivity) $x \sim y$ and $y \sim z$ implies $x \sim z$.

Then $Q = X/\sim$ denotes the set of equivalence classes of the relation. For $x \in X$, we denote the equivalence class containing x by $[x]$. There is a natural quotient map $q : X \to Q$ given by $q(x) = [x]$. We now define the *quotient topology* on Q such that $U \subseteq Q$ is open if $q^{-1}(U)$ is open in X. We call this the *quotient topology.*

(a) Show that the quotient topology is, in fact, a topology.

(b) Show that q is continuous if Q has the quotient topology.

(c) Show that the quotient topology is the finest topology (most open sets) such that q is continuous.

(d) The circle is a subset of $\mathbb{R}^2$ and so can be given the subspace topology. We can also think of the circle as the interval $[0, 2\pi]$ with the two endpoints identified. To be more precise, we define an equivalence relation by defining $0 \sim 2\pi$, and no other distinct points are equivalent. Since $[0, 2\pi]$ has a topology, we can consider the quotient topology on $[0, 2\pi]/\sim$. Show that $[0, 2\pi]/\sim$ is homeomorphic to the circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ with the subspace topology.

1-15. (final topology) If Y is a set, $(X, \mathcal{T})$ is a topological space, and $f : X \to Y$ is a function, then we can define a $\mathcal{T}'$ on Y by taking $\mathcal{T}'$ to be all subsets U of Y such that $f^{-1}(U) \in \mathcal{T}$.

(a) Show this defines a topology. This is sometimes called the *final topology* or *strong topology.* With this construction f is a continuous function. It is important to note that this construction works because of the set identities

$$\begin{aligned} f^{-1}(\cup_\alpha U_\alpha) &= \cup_\alpha f^{-1}(U_\alpha) \\ f^{-1}(\cap_\alpha U_\alpha) &= \cap_\alpha f^{-1}(U_\alpha) \end{aligned} \tag{1}$$

(b) Show that the final topology makes f a continuous function and that it is the finest topology (most open sets) that makes f a continuous function.

(c) Show that the quotient topology is the final topology for the quotient map $q(x) = [x]$.

1-16. (profinite topology) An arithmetic progress in $\mathbb{Z}$ is a set of the form $\{k + nl : n \in \mathbb{Z}\}$ where l is a positive integer and k is any integer. Define the *profinite topology* on $\mathbb{Z}$ in which the open sets are the empty set and unions of arithmetic progressions.

(a) Show that an arithmetic progression is also a closed set in this topology.

(b) Show that if there were only finitely many primes, then the set $\{-1, 1\}$ would be open.

(c) Then show that this set is not open and conclude that there are infinitely many primes.

(d) Let T^∞ be the product of countably infinitely many copies of the unit circle with the product topology. Define the map $\phi : \mathbb{Z} \to T^\infty$ as follows:

$$\phi(n) = (\exp(2\pi in/2), \exp(2\pi in/3), \exp(2\pi in/4), \exp(2\pi in/5), ...) .$$

Show that this map is injective and the induced topology on $\mathbb{Z}$ coincides with the profinite topology.

1.3 Properties of topologies

1-17. (first countable) $(X, \mathcal{T})$ is *first countable* if it has a local base $\mathcal{N}(x)$ such that at every point $x \in X$, the collection of neighborhoods $\mathcal{N}(x)$ is countable.

(a) Show that the metric topology on a metric space (X, d) is first countable.

(b) Is $(X, \mathcal{T})$ is first countable, show that every point $x \in X$ has a countable collection of open neighborhoods $U_n \ni x$ such that $U_{n+1} \subseteq U_n$, and x is an interior point for an open set O if and only if there is an index n such that $x \in U_n \subseteq O$.

(c) With the same definitions as the previous part, show that if you construct a sequence by picking arbitrary points $y_n \in U_n$, it follows that the sequence $\{y_n\}$ converges.

(d) If $(X, \mathcal{T})$ is first countable, and $(Y, \mathcal{S})$ is any topological space, show that $f : X \to Y$ is continuous if and only if it is sequentially continuous.

1-18. (second countable) A topological space is *second countable* if it has a countable basis.

(a) Show that $\mathbb{R}$ with the usual topology is second countable.

(b) Give an example of a space which is first countable but not second countable.

(c) Show that every second countable space is also first countable.

1-19. (dense/separable) A subspace A of X is called *dense* if the closure of A is X. A topological space X is called *separable*, if there exists a countable dense subset.

(a) Show that $\mathbb{R}^n$ is separable.

(b) Show that if X is second countable, then X is separable.

1-20. *(Hausdorff/T2) A space is *Hausdorff* (or *T2*) if for every two points $x, y \in X$, there are disjoint open sets U and V such that $x \in U$, $y \in V$.

(a) Show that a subspace of a Hausdorff space is Hausdorff but the quotient of a Hausdorff space may not be Hausdorff.

(b) Show that finite point sets in Hausdorff spaces are closed.

(c) (line with two origins) The line with two origins is defined to be the quotient of $\mathbb{R}\times \{1\} \cup \mathbb{R}\times \{2\}$ by the equivalence relation $(a, 1) \sim (a, 2)$ if $a \neq 0$. Show that the line with two origins is not Hausdorff.

(d) (Zariski topology) Consider the topology on $\mathbb{R}^n$ in which the open sets are the empty set and the complements of the common zero levels sets of finitely many polynomials. Show that this is indeed a topology on R^n. This is called the *Zariski topology.* Show also that the Zariski topology is not Hausdorff.

1-21. (T1) A topological space X is called a *T1-space* (or a Tychonoff space) if for any two different points $x, y \in X$ there exists an open set U which contains x but does not contain y.

(a) Prove that a space X is a T1-space if and only if any subset consisting of a single point is closed.

(b) Let the group $\mathbb{R}$ act on $\mathbb{R}^2$ by

$$t.(x, y) = (x, y + tx).$$

Prove that the quotient space with the quotient topology is not Hausdorff, but is the union of two disjoint Hausdorff subspaces. Also show that the quotient space is a T1-space.

1-22. Find a topological space X and a sequence x_n in X which converges but has more than one limit. What additional property on X implies that limits of sequences are unique?

1-23. A function $f : \mathbb{R} \to \mathbb{R}$ is *lower semi continuous* if for all $x \in \mathbb{R}, \epsilon > 0$, there exists a $\delta > 0$ such that $|y - x| < \delta$ implies that $f(y) > f(x) - \epsilon$.

(a) Show that the collection $\mathcal{B} = \{(\alpha, \infty) \mid \alpha \in \mathbb{R}\}$ is a basis. Let T' denote the topology generated by $\mathcal{B}$. Show that $T' \subset T_{metric}$ and the containment is strict. (Hint: One idea is to show that T' is not Hausdorff.)

(b) Show that the collection $\mathcal{B}$ along with the empty set and all of $\mathbb{R}$ is the topology generated by the base $\mathcal{B}$, *i.e.* $T' = \mathcal{B} \cup \{\emptyset, \mathbb{R}\}$.

(c) Show that T' is second countable.

(d) Show that a function $f : (\mathbb{R}, T_{metric}) \to (\mathbb{R}, T')$ is continuous, if and only if it is lower semi-continuous by the earlier definition.

(e) Show that a function $f : (\mathbb{R}, T') \to (\mathbb{R}, T_{metric})$ is continuous, if and only if it is a constant function.

1-24. Let $X = \mathbb{N} \cup \{e\}$. Define a collection $\mathcal{T}$ by $A \subseteq X$ is in $\mathcal{T}$ if and only if A does not contain e (this includes the empty set) or both $e \in A$ and A^c is finite (this includes X).

(a) Show that $\mathcal{T}$ is a topology on X.

(b) Show that $\mathcal{T}$ is second countable.

(c) Show that $\mathbb{N}$ is dense in $(X, \mathcal{T})$.

(d) Is $(X, \mathcal{T})$ compact? (see later section)

(e) A function $f : \mathbb{N} \to \mathbb{R}$ is the same thing as a sequence x_n. We will say that $g : X \to \mathbb{R}$ is a continuous extension of f if $g(n) = f(n) \, \forall n \in \mathbb{N}$. Show that f has a continuous extension iff $x_n = f(n)$ is a convergent sequence. Further, the continuous extension is given by $g(e) = \lim_{n\to\infty} f(n)$.

(f) Every element $a = (l, a_1, a_2, a_3, \ldots) \in \mathbb{R} \times \mathbb{R}^N$ defines a function $f_a : X \to \mathbb{R}$ by $f_a(n) = a_n, f_a(e) = l$. Let $Y \subset \mathbb{R} \times \mathbb{R}^N$ denote the set of all the convergent sequences with their associated limits, *i.e.* $(l, a_1, a_2, a_3, \ldots) \in Y \implies a_n \to l$. Find the weakest topology on X such that for all $a \in Y$, $f_a : X \to \mathbb{R}$ is continuous.

(g) Can you find a metric on X such that the metric topology is identical to the topology $\mathcal{T}$ above? Any topology with this property is said to be *metrizable*.

2 Problems on compactness

2-1. Prove or disprove:

(a) Show that $(0, 1]$ is not compact.

(b) A is finite and U is a open subset of $\mathbb{R}$. If $A \subseteq U$, there exists an $\epsilon > 0$ such that for all $x \in A$, $B(x, \epsilon) \subseteq U$. ($B(x, \epsilon)$ denotes the open ball of radius ϵ centered at x.)

(c) P is countable and U is a open subset of $\mathbb{R}$. If $P \subseteq U$, there exists an $\epsilon > 0$ such that for all $x \in P$, $B(x, \epsilon) \subseteq U$.

(d) F is closed and U is a open subset of $\mathbb{R}$. If $F \subseteq U$, there exists an $\epsilon > 0$ such that for all $x \in F$, $B(x, \epsilon) \subseteq U$.

(e) K is compact and U is a open subset of $\mathbb{R}$. If $K \subseteq U$, there exists an $\epsilon > 0$ such that for all $x \in K$, $B(x, \epsilon) \subseteq U$.

2-2. Exampes

(a) Any finite topological space is compact.

(b) The finite-dimensional sphere $\left\{x \in \mathbb{R}^n : |x|^2 = 1\right\}$ is compact.

(c) The Cantor set K defined by $K = [0, 1] \setminus \bigcup_{m=1}^{\infty} \bigcup_{k=0}^{3^{m-1}-1} \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$ is compact.

(d) A set X is has the discrete topology if for each $x \in X$, the set $\{x\}$ is open. Show that there is only one such topology and that it is compact if and only if X is finite

(e) Let $\mathbb{RP}^n$ denote the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$, by the equivalence relation $x \sim y$ iff $\exists \lambda \neq 0$, s.t. $x = \lambda y$. Show that $\mathbb{RP}^n$ is compact.

(f) Given a set X, the finite complement topology is the topology where open sets are sets whose complement are finite. Show this is a topology and that X is compact with this topology.

2-3. Prove or disprove the following:

2-4. Suppose $(X, \mathcal{T})$ is a topological space.

(a) **We can define compactness of a subset as follows: $F \subseteq X$ is compact if every collection $\{U_i\}_{i \in I}$ of open sets in X such that $F \subseteq \bigcup_{i \in I} U_i$, there is a finite subcover. Show that the topological space F with the subspace topology is compact if and only if F is compact as a subset of X.

(b) Let $S_2 \subset S_1 \subset X$. We give S_1 the subspace topology.

i. Show that if S_2 is open in S_1 then it need not be open in X. Show that it is if S_1 is open in X.

ii. Show that if S_2 is compact in S_1, then S_2 is compact in X.\

(c) *Show that if X is compact and $F \subset X$ is closed, then F is compact (in the subspace topology).

2-5. Proof of Extreme Value Theorem

(a) Prove $[0, 1]$ is compact.

(b) Prove that if X and Y are compact, then $X \times Y$ is compact.

(c) Prove the Heine Borel Theroem (for compact subsets of $\mathbb{R}^n$).

(d) If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact (with the subspace topology).

(e) Prove the Extreme Value Theorem.

2-6. We look at how compactness and closedness are related.

(a) Show that compact subsets need not be closed. Hint: consider the indiscrete topology that has only two open sets.

(b) Show that compact subsets of a Hausdorff space are closed.

2-7. (sequential compactness) A sequence $\{x_n\}$ in a topological space X *converges* to $x_\infty \in X$ if for every open set U containing x_∞, there exists N such that $x_n \in U$ if $n > N$. We say a space is *sequentially compact* if every sequence has a convergent subsequence. Prove that in a metric space, a set is compact if and only if it is sequentially compact

2-8. Show that if X is compact and Y is Hausdorff and $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.

2-9. (local compactness/one point compactification) A space is *locally compact* if every point has a compact neighborhood. (A neighborhood of a point x is an set N such that there exists and open set U such that $x \in U \subseteq N$.)

(a) If X is a compact space, then show that X is locally compact.

(b) Show that the rational numbers $\mathbb{Q}$ is not locally compact. (Hint: Given a neighborhood of $q \in \mathbb{Q}$, it must contain a closed interval $[q-r, q+r] \cap \mathbb{Q}$ which is compact if N is compact. Then show that $[q-r, q+r] \cap \mathbb{Q}$ has a cover with no finite subcover.)

(c) There is a canonical way to add one point to a locally compact Hausdorff space to get a compact space. Namely, if X is locally compact Hausdorff, let $\bar{X} = X \cup \{\infty\}$. The open sets of $\bar{X}$ are the open sets of X together with the sets $(X \setminus K) \cup \{\infty\}$, where K is a compact subset of X. Prove that $\bar{X}$, called the *one point compactification* of X, is a compact Hausdorff space.

(d) Show that the one point compactification of $\mathbb{R}^n$ is homeomorphic to the sphere S^n.

2-10. A sequence $\{x_n\}$ in a metric space (X, d) is said to be *Cauchy* if for every $\varepsilon > 0$ there exists $N > 0$ such that $d(x_n, x_m) < \varepsilon$ whenever $n, m \geq N$. A metric space is *complete* if every Cauchy sequence converges.

(a) Show that every convergent sequence is Cauchy. (The converse is only true if the metric space is complete.)

(b) Give an example of a metric space that is not complete.

(c) Show that a compact metric space can be covered by finitely many balls of any given radius.

(d) Show that every compact metric space is complete.

(e) For $x, y \in \mathbb{R}$ let

$$d(x,y) = \frac{|x-y|}{1+|x-y|}.$$

Prove that with this metric, the entire space of $\mathbb{R}$ is not compact even though it is closed and bounded.

(f) A metric space (X, d) is totally bounded if and only if for every real number $\varepsilon > 0$, there exists a finite collection of open balls in X of radius ε whose union contains X. Show that a complete, totally bounded metric space is compact. Explain why the previous example fails to satisfy these assumptions.

3 Problems on connectedness

3-1. Basic examples

(a) Show $(0, 1)$ is connected.

(b) Show $(0, 2) \setminus \{1\}$ is disconnected.

(c) Show that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic. Hint: use the notion of a connected set.

(d) Find all the different topologies, up to homeomorphism, on a 4-element set, which make it a connected topological space.

(e) Prove that the closure of a connected subspace is connected, but if the closure of a space is connected the space may not be connected.

3-2. Intermediate Value Theorem.

(a) Prove that if X is connnected and $f : X \to Y$ is continuous then $f(X)$ is connected.

(b) Prove the Intermediate Value Theorem.

(c) Use the Intermediate Value Theorem to prove a special case of Brouwer's Fixed Point Theorem: every continuous map $f : [-1, 1] \to [-1, 1]$ has a fixed point.
Answer: Consider $g(x) = x - f(x)$. $g(-1) \leq 0$ and $g(1) \geq 0$.

3-3. (path connected). A *path* in X is a continuous map $\gamma : [a, b] \to X$. A space X is *path connected* if any two points can be joined by a path.

(a) Show that if X is path connected, then it is connected.

(b) Sow that if $f : X \to Y$ is continuous and X is path connected then $f(X)$ is path connected.

(c) Let X be the union of the origin in $\mathbb{R}^2$ and the graph of $\sin(1/x)$ on $(0, \infty)$. Show X is connected but not path connected.

(d) A space X is called locally path-connected, if for each $x \in X$ and every neighborhood U of x, there exists a path-connected neighborhood V of x contained in U. Show that if X is connected and locally path-connected, then it is path-connected.

(e) Show that every open subset of $\mathbb{R}^n$ is locally path connected.

(f) Recall the Cantor set K is defined by

$$K = [0, 1] \setminus \bigcup_{m=1}^{\infty} \bigcup_{k=0}^{3^{m-1}-1} \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m} \right).$$

Show that the complement of $K \times K$ in the unit square $[0, 1] \times [0, 1]$ is path-connected.

3-4. (connected component) Given X, we can define an equivalence relation on X by setting $x \sim y$ if there is a connected subset containing both x and y. The equivalence classes are called *components* or *connected components* of X.

(a) Prove that each connected component of a topological space X is closed.

(b) Show that if A is a both open and closed, non-empty, connected subset of a topological space X, then A is a connected component.

(c) Show that if a topological space has finitely many connected components, then each of them is open and closed.